A symmetric structure of variational and adjoint systems of stochastic Hamiltonian systems

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***Abstract*— The authors have extended deterministic port-Hamiltonian systems into stochastic dynamical systems which are described by stochastic differential equations written in the sense of Itô, called stochastic port-Hamiltonian systems. This paper introduces variational systems and their adjoint ones for the stochastic port-Hamiltonian systems. We also reveal some of their properties, particularly an extension of a self-adjoint property of deterministic Hamiltonian systems, which plays an important role in learning optimal control for the deterministic Hamiltonian systems.**

I. INTRODUCTION

Physical systems are practically important and they have good properties for the control design such as passivity, symmetry and so on. Hamiltonian systems [1], [2], [3] have been introduced to represent physical systems and they explicitly possess such characteristics. A novel iterative learning control method based on variational symmetry of Hamiltonian systems was proposed in [4] and it allows one to solve not only trajectory tracking control problem but also a class of optimal control problems by iteration of laboratory experiments. Under certain conditions, a state-space realization of the adjoint of the variational system for a Hamiltonian system coincides with a time-reversal version of that of the variational system. This symmetric property is called variational symmetry. Thanks to variational symmetry, this method does not require the precise knowledge of the plant model and it can deal with infinite dimensional optimal control problems, more concretely, optimal control problems on L_2 signal space is considered. Although this method works well for some control problems, many conventional results of iterative learning control, e.g. [5], [4], assume an ideal experimental environment, that is, any environmental disturbance and measurement noise during experiments are not considered so far. Moreover, standard disturbance attenuation methods, e.g. robust and adaptive control methods, are not available, because we do not know the plant model.

The authors consider the plant system with the above uncertainties as a stochastic system and we focus on stochastic control theory to take disturbances during experiments into account. The dynamics of a stochastic system is described by a stochastic differential equation [6] and its solution is a random process, so this system inherently has uncertainty. In our previous result [7], [8], we have introduced stochastic Hamiltonian systems which are extension of deterministic Hamiltonian systems to stochastic one and have proposed a stabilization framework based on stochastic passivity [9] and pairs of coordinate and feedback transformations preserving the stochastic Hamiltonian structure named stochastic generalized canonical transformations.

The main purpose of this paper is to investigate a stochastic symmetric property of stochastic Hamiltonian systems which corresponds to variational symmetry for deterministic Hamiltonian systems. Firstly, we will define the variational systems of a class of general nonlinear stochastic systems. Then their adjoint systems will be defined based on systems utilized in the stochastic maximum principle in stochastic optimal control, e.g. [10]. Here let us note that the stochastic adjoint system is a nontrivial extension of the deterministic one, e.g. [1], [11]. An input-output relation between the variational and its adjoint systems will be shown. Secondly, stochastic Hamiltonian systems will be focused on. We will show that the variational system of a stochastic Hamiltonian system is described by another linear one. Then we will derive a condition under which the adjoint system coincides with the time-reversal version of the variational one. It will be proven that this property is an extension of variational symmetry of deterministic Hamiltonian systems.

In the sequel, we utilize the notation $\partial_{(\cdot)}$ in order to denote the partial differential with respect to $(\cdot)$. For example, for a smooth function $f(x,t)$ on some Banach space, the following holds

$$\partial_x f(x,t) = f(x+\delta x, t) - f(x,t) + o(\|\delta x\|),$$

where

$$\lim_{\|\delta x\|\to 0} \frac{o(\|\delta x\|)}{\|\delta x\|} = 0.$$

II. VARIATIONAL SYSTEMS AND THEIR ADJOINT ONES FOR NONLINEAR STOCHASTIC SYSTEMS

In this section, we consider a class of general nonlinear stochastic systems described by the following stochastic differential equation written in the sense of Itô:

$$\begin{cases} \mathrm{d}x &= f(x,u,t)\,\mathrm{d}t + h(x,t)\,\mathrm{d}w \\ y &= s(x,u,t) \end{cases} . \quad (1)$$

Here $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n, h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times r}$ and $s : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ are sufficiently differentiable functions and they satisfy $f(O_{n1}, O_{m1}, t) = O_{n1}$, $h(O_{n1}, t) = O_{nr}$ and $s(O_{n1}, O_{m1}, t) = O_{m1}$, respectively, where O_{ij} denotes the $i \times j$ zero matrix. $w(t) \in \mathbb{R}^r$ is a standard Wiener

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process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is a sample space, $\mathcal{F}$ is the sigma algebra of observable random events and $\mathcal{P}$ is a probability measure on Ω. In what follows, we suppose that $f(x,u,t)$ and $h(x,t)$ satisfy reasonable sufficient conditions for the local existence and uniqueness of the solutions. For the details on such conditions, see [6], [12].

This section defines a variational system of a class of general nonlinear stochastic systems in a manner analogous to the case of deterministic systems [1], [4]. Then its adjoint system is defined based on a system utilized in the stochastic maximum principle [13], [14] in stochastic optimal control, e.g., [15], [14], [10] and references therein. Finally, an input-output relation between both systems is shown.

Let $(x(t,\epsilon), u(t,\epsilon), y(t,\epsilon)),\quad t \in [0,t^1]$ denote a family of state-input-output trajectories of the system in Eq. (1) parameterized by ϵ, such that $x(t,0) = x(t), u(t,0) = u(t), y(t,0) = y(t),\quad t \in [0,t^1]$. We define the variational system of (1) as the system in which the following quantities

$$x_v(t) = \frac{\partial x(t,0)}{\partial \epsilon}\ , \quad u_v(t) = \frac{\partial u(t,0)}{\partial \epsilon}\ , \quad y_v(t) = \frac{\partial y(t,0)}{\partial \epsilon}$$

are governed. In order to derive a stochastic differential equation which the variational process x_v satisfies, let us calculate the differentiation of $x(t,\epsilon)$ with respect to the time t. We have

$$\begin{aligned}
&\partial_t x(t,\epsilon)\\
&= f(x(t,\epsilon),u(t,\epsilon),t)\,\mathrm{d}t + h(x(t,\epsilon),t)\,\mathrm{d}w\\
&= f(x(t,0),u(t,0),t)\,\mathrm{d}t + \left\{\frac{\partial f(x(t,0),u(t,0),t)}{\partial x}\frac{\partial x(t,0)}{\partial \epsilon}\right.\\
&\quad \left. + \frac{\partial f(x(t,0),u(t,0),t)}{\partial u}\frac{\partial u(t,0)}{\partial \epsilon}\right\}\epsilon\,\mathrm{d}t + h(x(t,0),t)\,\mathrm{d}w\\
&\quad + \sum_{i=1}^{r}\frac{\partial h_i(x(t,0),t)}{\partial x}\frac{\partial x(t,0)}{\partial \epsilon}\epsilon\,\mathrm{d}w^i + o(|\epsilon|)\\
&= f(x(t),u(t),t)\,\mathrm{d}t + \left\{\frac{\partial f(x(t),u(t),t)}{\partial x}x_v\right.\\
&\quad \left. + \frac{\partial f(x(t),u(t),t)}{\partial u}u_v\right\}\epsilon\,\mathrm{d}t + h(x(t),t)\,\mathrm{d}w\\
&\quad + \sum_{i=1}^{r}\frac{\partial h_i(x(t),t)}{\partial x}x_v\epsilon\,\mathrm{d}w^i + o(|\epsilon|). \qquad (2)
\end{aligned}$$

From Eq. (2), the following stochastic differential equation is obtained which x_v should satisfy:

$$\begin{aligned}
&\mathrm{d}x_v\\
&= \lim_{\epsilon\to 0}\frac{\partial_t x(t,\epsilon) - \partial_t x(t,0)}{\epsilon}\\
&= \frac{\partial f(x,u,t)}{\partial x}x_v\,\mathrm{d}t + \frac{\partial f(x,u,t)}{\partial u}u_v\,\mathrm{d}t + \sum_{i=1}^{r}\frac{\partial h_i(x,t)}{\partial x}x_v\,\mathrm{d}w^i.
\end{aligned} \qquad (3)$$

The following equation can be obtained in a similar manner to Eq. (2),

$$y_v = \frac{\partial s(x,u,t)}{\partial x}x_v + \frac{\partial s(x,u,t)}{\partial u}u_v. \qquad (4)$$

Definition 1: Consider the system (1). The variational system of the system (1) is defined as

$$\left\{\begin{array}{rcl}
\mathrm{d}x &=& f(x,u,t)\,\mathrm{d}t + h(x,t)\,\mathrm{d}w\ , \quad x(0) = x_0\\
\mathrm{d}x_v &=& \dfrac{\partial f(x,u,t)}{\partial x}x_v\,\mathrm{d}t + \dfrac{\partial f(x,u,t)}{\partial u}u_v\,\mathrm{d}t\\
&& + \displaystyle\sum_{i=1}^{r}\frac{\partial h_i(x,t)}{\partial x}x_v\,\mathrm{d}w^i, \quad x_v(0) = x_{v,0}\\
y_v &=& \dfrac{\partial s(x,u,t)}{\partial x}x_v + \dfrac{\partial s(x,u,t)}{\partial u}u_v
\end{array}\right. . \qquad (5)$$

In this paper, we define an adjoint system of the variational system (5) based on a system utilized in the stochastic maximum principle in stochastic optimal control [14], [10].

Definition 2: Consider the system (1). The variational adjoint system of the system (1) is defined as

$$\left\{\begin{array}{rcl}
\mathrm{d}x &=& f(x,u,t)\,\mathrm{d}t + h(x,t)\,\mathrm{d}w\ , \quad x(0) = x_0\\
\mathrm{d}x_a &=& -\dfrac{\partial f(x,u,t)}{\partial x}^{\top}x_a\,\mathrm{d}t - \dfrac{\partial s(x,u,t)}{\partial x}^{\top}u_a\,\mathrm{d}t\\
&& -\displaystyle\sum_{i=1}^{r}\frac{\partial h_i(x,t)}{\partial x}^{\top}z_i\,\mathrm{d}t + \sum_{i=1}^{r}z_i\,\mathrm{d}w^i,\ x_a(t^1) = x_{a,t^1}\\
y_a &=& \dfrac{\partial f(x,u,t)}{\partial u}^{\top}x_a + \dfrac{\partial s(x,u,t)}{\partial u}^{\top}u_a
\end{array}\right. , \qquad (6)$$

where $(x_a(t), z(t)) \in \mathbb{R}^n \times \mathbb{R}^{n\times r}$ is a pair of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes [1].

Let us note that the variational adjoint equation is to be solved backwards, since the terminal condition x_{a,t^1} is given. However, the solution $(x_a(t), z(t))$ is required to be $\{\mathcal{F}_t\}_{t\geq 0}$-adapted. Any pair of processes $(x_a(t), z(t))$ satisfying (6) is called an adapted solution. For the detail of the backward stochastic differential equations, see [15], [16], [10].

Here let us prove the following theorem with respect to an input-output relation between the variational and its adjoint systems, which is an extension in the deterministic case.

Theorem 1: Consider the variational system (5) *and its adjoint system* (6) *on a time interval* $t \in [0,t^1]$. *Suppose that a pair of processes* (x_a, z) *is an adapted solution of the adjoint system* (6) *and that the initial condition of the variational system* $x_{v,0}$ *and the terminal state of the adjoint system* x_{a,t^1} *are the stochastic variables satisfying*

$$x_{v,0} = 0, \quad x_{a,t^1} = 0. \qquad (7)$$

Then the following relation holds:

$$E\left[\int_0^{t^1} y_a(t)^{\top}u_v(t)\,\mathrm{d}t\right] = E\left[\int_0^{t^1} u_a(t)^{\top}y_v(t)\,\mathrm{d}t\right]. \quad (8)$$

Here $E[\cdot]$ *denotes the expectation with respect to the probability measure* $\mathcal{P}$.

[1]A process $x(t)$ is called $\{\mathcal{F}_t\}_{t\geq 0}$-adapted if for each $t \geq 0$ the function $x(t)$ is $\mathcal{F}_t$-measurable.

Proof: Let us calculate the time variation of $x_a(t)^\top x_v(t)$ with the Itô formula as

$$\mathrm{d}(x_a^\top x_v) = \mathrm{d}x_a^\top x_v + x_a^\top \,\mathrm{d}x_v + \frac{1}{2}\,\mathrm{tr}\Bigg\{\frac{\partial^2 (x_a^\top x_v)}{\partial(x_v,x_a)^2} \times \begin{pmatrix}\mathcal{D}_x h_1 x_v,\cdots,\mathcal{D}_x h_d x_v\\ z_1,\cdots,z_r\end{pmatrix}\begin{pmatrix}\mathcal{D}_x h_1 x_v,\cdots,\mathcal{D}_x h_d x_v\\ z_1,\cdots,z_r\end{pmatrix}^\top\Bigg\}\,\mathrm{d}t, \tag{9}$$

where $\mathcal{D}_{(\cdot)}$ denotes the differential operator with respect to $(\cdot)$ and the following notation is utilized

$$\frac{\partial^2 (x_a^\top x_v)}{\partial(x_v,x_a)^2} := \frac{\partial}{\partial(x_v,x_a)}\left(\frac{\partial(x_a^\top x_v)}{\partial(x_v,x_a)}\right)^\top .$$

By substituting Eqs. (5) and (6) for Eq. (9), we have

$$\begin{aligned}
\mathrm{d}\,(x_a^\top x_v) &= \left(\left(-\left(\frac{\partial f}{\partial x}^\top x_a + \frac{\partial s}{\partial x}^\top u_a + \sum_{i=1}^r \frac{\partial h_i}{\partial x}^\top z_i\right)\mathrm{d}t + \sum_{i=1}^r z_i\,\mathrm{d}w^i\right)^\top x_v + x_a^\top\left(\frac{\partial f}{\partial x}x_v\,\mathrm{d}t + \frac{\partial f}{\partial u}u_v\,\mathrm{d}t + \sum_{i=1}^r \frac{\partial h_i}{\partial x}x_v\,\mathrm{d}w^i\right)\right.\\
&\quad + \frac{1}{2}\,\mathrm{tr}\left\{\sum_{i=1}^r z_i x_v^\top \frac{\partial h_i}{\partial x}^\top\right\}\mathrm{d}t + \frac{1}{2}\,\mathrm{tr}\left\{\sum_{i=1}^r \frac{\partial h_i}{\partial x}x_v z_i^\top\right\}\mathrm{d}t\\
&= \left(x_a^\top \frac{\partial f}{\partial u}u_v - u_a^\top \frac{\partial s}{\partial x}x_v\right)\mathrm{d}t - \sum_{i=1}^r z_i^\top \frac{\partial h_i}{\partial x}x_v\,\mathrm{d}t + \sum_{i=1}^r\left(z_i^\top + \frac{\partial h_i}{\partial x}\right)x_v\,\mathrm{d}w^i + \mathrm{tr}\left\{\sum_{i=1}^r \frac{\partial h_i}{\partial x}x_v z_i^\top\right\}\mathrm{d}t\\
&= \left(\left(y_a^\top - u_a^\top\frac{\partial s}{\partial u}\right)u_v - u_a^\top\left(y_v - \frac{\partial s}{\partial u}u_v\right)\right)\mathrm{d}t - \sum_{i=1}^r z_i^\top \frac{\partial h_i}{\partial x}x_v\,\mathrm{d}t + \sum_{i=1}^r\left(z_i^\top + \frac{\partial h_i}{\partial x}\right)x_v\,\mathrm{d}w^i + \mathrm{tr}\left\{\sum_{i=1}^r z_i^\top \frac{\partial h_i}{\partial x}x_v\right\}\mathrm{d}t\\
&= (y_a^\top u_v - u_a^\top y_v)\,\mathrm{d}t + \sum_{i=1}^r\left(z_i^\top + \frac{\partial h_i}{\partial x}\right)x_v\,\mathrm{d}w^i
\end{aligned}$$

The integration on $t\in[0,t^1]$ and the expectation yield

$$\begin{aligned}
&E[x_a(t^1)^\top x_v(t^1)] - E[x_a(0)^\top x_v(0)]\\
&\quad = E\left[\int_0^{t^1}(y_a^\top u_v - u_a^\top y_v)\,\mathrm{d}t\right] + E\left[\int_0^{t^1}\sum_{i=1}^r\left(z_i^\top + \frac{\partial h_i}{\partial x}\right)x_v\,\mathrm{d}w^i\right].
\end{aligned} \tag{10}$$

Since solutions $x_v(t)$ and $(x_a(t), z(t))$ are adapted processes and $h(x)$ is a sufficiently differentiable function, the property of the Itô integral, e.g. [6], [12], yields

$$E\left[\int_0^{t^1}\sum_{i=1}^r\left(z_i^\top + \frac{\partial h_i}{\partial x}\right)x_v\,\mathrm{d}w^i\right] = 0. \tag{11}$$

Equation (8) follows from Eqs. (7), (10) and (11). This proves the theorem. ■

III. Variational and its Adjoint Systems of Stochastic Hamiltonian Systems

This section investigates some properties of variational systems and their adjoint ones of stochastic Hamiltonian systems [7], [8]. Let us consider the following stochastic Hamiltonian system described as

$$\begin{cases}
\mathrm{d}x = (J(x,t) - R(x,t))\,\dfrac{\partial H(x,u,t)}{\partial x}^\top \mathrm{d}t + h(x,t)\,\mathrm{d}w\\
\qquad ,x(0) = x_0\\
\;\; y \;= -\dfrac{\partial H(x,u,t)}{\partial u}^\top .
\end{cases} \tag{12}$$

As in the case of the system in Eq. (1), $x(t) \in \mathbb{R}^n, u(t), y(t) \in \mathbb{R}^m$ describe the state, the input and the output, respectively and $w(t) \in \mathbb{R}^r$ is a standard Wiener process defined on a probability space $(\Omega,\mathcal{F},\mathcal{P})$. The structure matrix $J(x,t) \in \mathbb{R}^{n\times n}$ and the dissipation matrix $R(x,t) \in \mathbb{R}^{n\times n}$ are skew-symmetric and symmetric positive semi-definite for all x and t, respectively. A sufficiently differentiable function $h(x,t) \in \mathbb{R}^{n\times r}$ represents the noise port. We suppose that Hamiltonian $H(x,u,t)$ is a sufficiently differentiable function and $h(O_{n1},t) = O_{nr}$. Moreover, $h(x,t)$ satisfies reasonable sufficient conditions for the local existence and uniqueness of the solutions.

Firstly, we show a property of the variational system of the stochastic Hamiltonian system (12).

Theorem 2: Consider the stochastic Hamiltonian system in (12). *Suppose that J and R are constant matrices. Then the variational system of the system* (12) *is described by another linear stochastic Hamiltonian system:*

$$\begin{cases}
\mathrm{d}x \;=\; (J(x,t) - R(x,t))\,\dfrac{\partial H(x,u,t)}{\partial x}^\top \mathrm{d}t + h(x,t)\,\mathrm{d}w\\
\qquad ,x(0) = x_0\\
\mathrm{d}x_v \;=\; (J - R)\,\dfrac{\partial H_v(x,u,x_v,u_v t)}{\partial x_v}^\top \mathrm{d}t\\
\qquad + \displaystyle\sum_{i=1}^r \frac{\partial h_i(x,t)}{\partial x}x_v\,\mathrm{d}w^i \quad , \quad x_v(0) = x_{v,0}\\
y_v \;=\; -\dfrac{\partial H_v(x,u,x_v,u_v,t)}{\partial u_v}^\top ,
\end{cases} \tag{13}$$

where a Hamiltonian is given by

$$H_v(x,u,x_v,u_v,t) = \frac{1}{2}\begin{pmatrix}x_v\\u_v\end{pmatrix}^\top \frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\begin{pmatrix}x_v\\u_v\end{pmatrix}.$$

Proof: This theorem can be proven in a similar manner as Theorem 1 in [4]. According to the definition of the variational system in (3), the variational system of the stochastic Hamiltonian system (12) is obtained as

$$\begin{aligned}
\mathrm{d}x_v &= \frac{\partial}{\partial x}\left((J-R)\frac{\partial H}{\partial x}^\top\right)x_v\,\mathrm{d}t \\
&\quad + \frac{\partial}{\partial u}\left((J-R)\frac{\partial H}{\partial x}^\top\right)u_v\,\mathrm{d}t + \sum_{i=1}^r \frac{\partial h_i}{\partial x}x_v\,\mathrm{d}w^i \\
&= (J-R,\ O_{nn})\frac{\partial^2 H}{\partial(x,u)^2}\begin{pmatrix}x_v\\u_v\end{pmatrix}\mathrm{d}t + \sum_{i=1}^r \frac{\partial h_i}{\partial x}x_v\,\mathrm{d}w^i \\
&= (J-R,\ O_{nn})\left(\frac{\partial}{\partial(x_v,u_v)}\left\{\frac{1}{2}\begin{pmatrix}x_v\\u_v\end{pmatrix}^\top\frac{\partial^2 H}{\partial(x,u)^2}\right.\right. \\
&\quad \left.\left.\times\begin{pmatrix}x_v\\u_v\end{pmatrix}\right\}\right)^\top\mathrm{d}t + \sum_{i=1}^r \frac{\partial h_i}{\partial x}x_v\,\mathrm{d}w^i \\
&= (J-R)\frac{\partial H_v}{\partial x_v}^\top\mathrm{d}t + \sum_{i=1}^r \frac{\partial h_i}{\partial x}x_v\,\mathrm{d}w^i. \qquad (14)
\end{aligned}$$

Then let us calculate the output y_v from Eq. (4) in a similar calculation as Eq. (14) as

$$\begin{aligned}
y_v &= -\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u}\right)^\top x_v + \frac{\partial}{\partial u}\left(\frac{\partial H}{\partial u}\right)^\top u_v \\
&= (O_{nn}, -I_n)\frac{\partial^2 H}{\partial(x,u)^2}\begin{pmatrix}x_v\\u_v\end{pmatrix} \\
&= (O_{nn}, -I_n)\left(\frac{\partial H_v}{\partial(x_v,u_v)}\right)^\top \\
&= -\frac{\partial H_v}{\partial u_v}^\top. \qquad (15)
\end{aligned}$$

Equations (14) and (15) prove the theorem. ■

Remark 1: Theorem 2 reduces to the corresponding result for the deterministic Hamiltonian systems in [4] as a special case.

Finally, we derive conditions under which an adjoint system of the stochastic Hamiltonian system (12) coincides with the time-reversal version of its variational one. This property is an extension of variational symmetry for the deterministic Hamiltonian systems in [4].

Theorem 3: Consider the stochastic Hamiltonian system in (12)*. Suppose that J and R are constant matrices and that $J-R$ is nonsingular. Furthermore, suppose that there exists a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ satisfying*

$$J = -TJT^{-1}, \quad R = TRT^{-1} \qquad (16)$$

$$\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2} = \begin{pmatrix}T & O_{nn}\\ O_{nn} & I_n\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\begin{pmatrix}T^{-1} & O_{nn}\\ O_{nn} & I_n\end{pmatrix} \qquad (17)$$

$$(J-R)T\frac{\partial h_i(x,t)}{\partial x}^\top T^{-1}(J-R)^{-1}\frac{\partial h_i(x,t)}{\partial x} = O_{nn}, \quad (i=1,2,\cdots,r). \qquad (18)$$

Then there exist the following processes $z_i,\ i=1,2,\cdots,r$ in the adjoint system in (6)

$$z_i = -T^{-1}(J-R)^{-1}\frac{\partial h_i(x,t)}{\partial x}(J-R)Tx_a, \qquad (19)$$

and the dynamics of the adjoint system of the stochastic Hamiltonian system coincides with the time-reversal version of its variational one in (13)*.*

Proof: This theorem can be proven in a similar manner as Theorem 1 in [4]. According to the definition of the adjoint system in (6), the adjoint system of the stochastic Hamiltonian system (12) is obtained as

$$\begin{aligned}
\mathrm{d}x_a &= -\frac{\partial}{\partial x}\left((J-R)\frac{\partial H}{\partial x}^\top\right)^\top x_a\,\mathrm{d}t + \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial u}^\top\right)^\top u_a\,\mathrm{d}t \\
&\quad - \sum_{i=1}^r \frac{\partial h_i}{\partial x}^\top z_i\,\mathrm{d}t + \sum_{i=1}^r z_i\,\mathrm{d}w^i \\
&= (-I_n,\ O_{nn})\left[\begin{pmatrix}J-R & O_{nn}\\ O_{nn} & -I_n\end{pmatrix}\frac{\partial^2 H}{\partial(x,u)^2}\right]^\top\begin{pmatrix}x_a\\u_a\end{pmatrix}\mathrm{d}t \\
&\quad - \sum_{i=1}^r \frac{\partial h_i}{\partial x}^\top z_i\,\mathrm{d}t + \sum_{i=1}^r z_i\,\mathrm{d}w^i. \qquad (20)
\end{aligned}$$

The output y_a can be calculated as

$$\begin{aligned}
y_a &= \frac{\partial}{\partial u}\left((J-R)\frac{\partial H}{\partial x}^\top\right)^\top x_a - \frac{\partial}{\partial u}\left(\frac{\partial H}{\partial u}\right)^\top u_a \\
&= (O_{nn},\ I_n)\left[\begin{pmatrix}J-R & O_{nn}\\ O_{nn} & -I_n\end{pmatrix}\frac{\partial^2 H}{\partial(x,u)^2}\right]^\top\begin{pmatrix}x_a\\u_a\end{pmatrix}. \qquad (21)
\end{aligned}$$

Here let us transform the system (20) by a coordinate transformation $\bar{x}_a = -(J-R)Tx_a$. Then the dynamics of the transformed system is calculated by the Itô formula as

$$\begin{aligned}
\mathrm{d}\bar{x}_a &= -(J-R)T\left((-I_n, O_{nn})\left[\begin{pmatrix}J-R & O_{nn}\\ O_{nn} & -I_n\end{pmatrix}\frac{\partial^2 H}{\partial(x,u)^2}\right]^\top\right. \\
&\quad \left.\times\begin{pmatrix}x_a\\u_a\end{pmatrix}\mathrm{d}t - \sum_{i=1}^r \frac{\partial h_i}{\partial x}^\top z_i\,\mathrm{d}t + \sum_{i=1}^r z_i\,\mathrm{d}w^i\right) \\
&\quad + \frac{1}{2}\begin{pmatrix}\mathrm{tr}\left\{\mathcal{D}^2_{x_a}(-(J-R)^1Tx_a)zz^\top\right\}\\ \vdots \\ \mathrm{tr}\left\{\mathcal{D}^2_{x_a}(-(J-R)^nTx_a)zz^\top\right\}\end{pmatrix}\mathrm{d}t \\
&= -(J-R, O_{nn})\begin{pmatrix}-T & O_{nn}\\ O_{nn} & -I_n\end{pmatrix}\frac{\partial^2 H}{\partial(x,u)^2}\begin{pmatrix}-(J+R) & O_{nn}\\ O_{nn} & -I_n\end{pmatrix} \\
&\quad \times\begin{pmatrix}x_a\\u_a\end{pmatrix}\mathrm{d}t + \sum_{i=1}^r(J-R)T\frac{\partial h_i}{\partial x}^\top z_i\,\mathrm{d}t - \sum_{i=1}^r(J-R)Tz_i\,\mathrm{d}w^i
\end{aligned}$$

$$
\begin{aligned}
&= -(J - R,\ O_{nn}) \begin{pmatrix} -T & O_{nn} \\ O_{nn} & -I_n \end{pmatrix} \frac{\partial^2 H}{\partial (x,u)^2} \begin{pmatrix} -T^{-1} & O_{nn} \\ O_{nn} & -I_n \end{pmatrix} \\
&\quad \times \begin{pmatrix} T(J+R) & O_{nn} \\ O_{nn} & I_n \end{pmatrix} \begin{pmatrix} x_a \\ u_a \end{pmatrix} \mathrm{d}t + \sum_{i=1}^{r} (J-R) T \frac{\partial h_i}{\partial x}^{\top} z_i \,\mathrm{d}t \\
&\quad - \sum_{i=1}^{r} (J-R) T z_i \,\mathrm{d}w^i \\
&= -(J - R,\ O_{nn}) \begin{pmatrix} T & O_{nn} \\ O_{nn} & I_n \end{pmatrix} \frac{\partial^2 H}{\partial (x,u)^2} \begin{pmatrix} T^{-1} & O_{nn} \\ O_{nn} & I_n \end{pmatrix} \\
&\quad \times \begin{pmatrix} -(J-R) T x_a & O_{nn} \\ O_{nn} & u_a \end{pmatrix} \mathrm{d}t + \sum_{i=1}^{r} (J-R) T \frac{\partial h_i}{\partial x}^{\top} z_i \,\mathrm{d}t \\
&\quad - \sum_{i=1}^{r} (J-R) T z_i \,\mathrm{d}w^i \\
&= -(J - R,\ O_{nn}) \begin{pmatrix} T & O_{nn} \\ O_{nn} & I_n \end{pmatrix} \frac{\partial^2 H}{\partial (x,u)^2} \begin{pmatrix} T^{-1} & O_{nn} \\ O_{nn} & I_n \end{pmatrix} \\
&\quad \times \begin{pmatrix} \bar{x}_a \\ u_a \end{pmatrix} \mathrm{d}t + \sum_{i=1}^{r} (J-R) T \frac{\partial h_i}{\partial x}^{\top} T^{-1} (J-R)^{-1} \frac{\partial h_i}{\partial x} \bar{x}_a \,\mathrm{d}t \\
&\quad - \sum_{i=1}^{r} \frac{\partial h_i}{\partial x} \bar{x}_a \,\mathrm{d}w^i.
\end{aligned}
$$

Here the fourth equality follows from

$$
\begin{aligned}
T(J+R) &= (TJT^{-1} + TRT^{-1})T \\
&= -(J-R)T
\end{aligned}
$$

with the condition (16) and the last equality follows from Eq. (19). The conditions (17) and (18) and Eq. (19) yield

$$
\begin{aligned}
\mathrm{d}\bar{x}_a = &-(J-R, O_{nn}) \frac{\partial^2 H(x,u,t)}{\partial (x,u)^2} \begin{pmatrix} \bar{x}_a \\ u_a \end{pmatrix} \mathrm{d}t \\
&- \sum_{i=1}^{r} \frac{\partial h_i(x,t)}{\partial x} \bar{x}_a \,\mathrm{d}w^i.
\end{aligned} \tag{22}
$$

It follows from Eq. (21) with a similar calculation, that

$$
\begin{aligned}
y_a &= -(O_{nn},\ I_n)\ \frac{\partial^2 H(x,u,t)}{\partial (x,u)^2} \begin{pmatrix} \bar{x}_a \\ u_a \end{pmatrix} \\
&= -\frac{\partial H_v(x,u,\bar{x}_a,u_a,t)}{\partial u_a}^{\top}.
\end{aligned} \tag{23}
$$

Theorem is immediately proven by comparing Eqs. (22) and (23) with the dynamics of the variational system in (13). ∎

Remark 2: In Theorem 3, conditions (16) and (17) are the same as those assumed in the corresponding result for the deterministic Hamiltonian systems in [4] and the other one (18) is required only for the stochastic Hamiltonian systems. In [4], it is shown that the following nonsingular matrix

$$
T = \begin{pmatrix} I_m & O_{mm} \\ O_{mm} & -I_m \end{pmatrix}
$$

satisfies the conditions (16) and (17) for the deterministic typical mechanical systems with the dimension $n = 2m$. Here let us check how strict the condition (18) is by a case study of the following typical mechanical system with noise, which we are mainly interested in for optimal gait generation of walking robots (see [17], [18])

$$
\begin{cases}
\begin{pmatrix} \mathrm{d}q \\ \mathrm{d}p \end{pmatrix} = \begin{pmatrix} O_{mm} & I_m \\ -I_m & -R_D \end{pmatrix} \begin{pmatrix} \dfrac{\partial H(q,p,u,t)}{\partial q}^{\top} \\ \dfrac{\partial H(q,p,u,t)}{\partial p}^{\top} \end{pmatrix} \mathrm{d}t \\
\qquad\qquad + \begin{pmatrix} O_{mm} \\ I_m \end{pmatrix} u \,\mathrm{d}t + \begin{pmatrix} O_{mr} \\ h_p(q,p,t) \end{pmatrix} \mathrm{d}w \\
y = -\dfrac{\partial H(q,p,u,t)}{\partial u}^{\top} = q
\end{cases} \tag{24}
$$

with the state $x := (q^\top, p^\top)^\top \in \mathbb{R}^{2m}$ and the Hamiltonian

$$
H(q,p,u,t) = \frac{1}{2} p^\top M(q)^{-1} p + V(q,t) - u^\top q. \tag{25}
$$

Here a positive definite matrix $M(q) \in \mathbb{R}^{m \times m}$ denotes the inertia matrix. A positive semi-definite matrix $R_D \in \mathbb{R}^{m \times m}$ denotes the friction coefficients and a scalar function $V(q,t) \in \mathbb{R}$ denotes the potential energy of the system. $w(t) \in \mathbb{R}^r$ denotes a standard Wiener process and $h_p(q,p,t) \in \mathbb{R}^{m \times r}$ represents the noise port.

Since we have

$$
\begin{aligned}
\frac{\partial h_i(q,p,t)}{\partial x} &= \begin{pmatrix} O_{mm} & O_{mm} \\ \frac{\partial h_{p,i}}{\partial q} & \frac{\partial h_{p,i}}{\partial p} \end{pmatrix} \\
(J-R)^{-1} &= \begin{pmatrix} -R_D & -I_m \\ I_m & O_{mm} \end{pmatrix},
\end{aligned}
$$

the condition (18) is reduced to

$$
\begin{aligned}
&(J-R) T \frac{\partial h_i}{\partial x}^{\top} T^{-1} (J-R)^{-1} \frac{\partial h_i}{\partial x} \\
&= \begin{pmatrix} O_{mm} & I_m \\ -I_m & -R_D \end{pmatrix} \begin{pmatrix} I_m & O_{mm} \\ O_{mm} & -I_m \end{pmatrix} \begin{pmatrix} O_{mm} & \frac{\partial h_{p,i}}{\partial q}^{\top} \\ O_{mm} & \frac{\partial h_{p,i}}{\partial p}^{\top} \end{pmatrix} \\
&\quad \times \begin{pmatrix} I_m & O_{mm} \\ O_{mm} & -I_m \end{pmatrix} \begin{pmatrix} -R_D & -I_m \\ I_m & O_{mm} \end{pmatrix} \begin{pmatrix} O_{mm} & O_{mm} \\ \frac{\partial h_{p,i}}{\partial q} & \frac{\partial h_{p,i}}{\partial p} \end{pmatrix} \\
&= \begin{pmatrix} O_{mm} & I_m \\ -I_m & -R_D \end{pmatrix} \begin{pmatrix} O_{mm} & \frac{\partial h_{p,i}}{\partial q}^{\top} \\ O_{mm} & -\frac{\partial h_{p,i}}{\partial p}^{\top} \end{pmatrix} \begin{pmatrix} -\frac{\partial h_{p,i}}{\partial q} & -\frac{\partial h_{p,i}}{\partial p} \\ O_{mm} & O_{mm} \end{pmatrix} \\
&= O_{2m2m}.
\end{aligned} \tag{26}
$$

Equation (26) implies that the condition (18) always holds for the typical mechanical system with noise in (24) with any noise port $h_p(q,p,t)$.

IV. Conclusion

This paper has introduced the variational and its adjoint systems for stochastic Hamiltonian systems and has revealed some of their properties. Firstly, we have defined the variational and its adjoint systems for a class of general nonlinear stochastic systems and have clarified their input-output relation which is an extension of that in deterministic systems. Secondly, we have shown that a variational system of a stochastic Hamiltonian system is described by another linear one. Finally, we have derived conditions under which the adjoint system coincides with the time-reversal version of the variational one. This property is an extension of variational symmetry of deterministic Hamiltonian systems, which plays an important role in learning optimal control proposed in [4] for the deterministic systems.

REFERENCES

[1] P. E. Crouch and A. J. van der Schaft, *Variational and Hamiltonian Control Systems*, ser. Lecture Notes on Control and Information Science. Berlin: Springer-Verlag, 1987, vol. 101.

[2] B. Maschke and A. J. van der Schaft, "Port-controlled Hamiltonian systems: modelling origins and system theoretic properties," in *Proc. 2nd IFAC Symp. Nonlinear Control Systems*, 1992, pp. 282–288.

[3] A. J. van der Schaft and B. M. J. Maschke, "Mathematical modeling of constrained Hamiltonian systems," in *Proc. 3rd IFAC Symp. Nonlinear Control Systems*, 1995.

[4] K. Fujimoto and T. Sugie, "Iterative learning control of Hamiltonian systems: I/O based optimal control approach," *IEEE Trans. Autom. Contr.*, vol. 48, no. 10, pp. 1756–1761, 2003.

[5] S. Arimoto, S. Kawamura, and F. Miyazaki, "Bettering operation of robotics," *J. Robotic Systems*, vol. 1, no. 2, pp. 123–140, 1984.

[6] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes*, 2nd ed. Amsterdam/Tokyo: North-Holland/Kodansha, 1989.

[7] S. Satoh and K. Fujimoto, "On passivity based control of stochastic port-Hamiltonian systems," in *Proc. 47th IEEE Conf. on Decision and Control*, 2008, pp. 4951–4956.

[8] ——, "Stabilization of time-varying stochastic port-Hamiltonian systems based on stochastic passivity," in *Proc. IFAC Symp. Nonlinear Control Systems*, 2010, pp. 611–616.

[9] P. Florchinger, "A passive system approach to feedback stabilization of nonlinear control stochastic systems," *SIAM J. Control Optim.*, vol. 37, no. 6, pp. 1848–1864, 1999.

[10] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*. New York: Springer-Verlag, 1999.

[11] K. Fujimoto, J. M. A. Scherpen, and W. S. Gray, "Hamiltonian realizations of nonlinear adjoint operators," *Automatica*, vol. 38, no. 10, pp. 1769–1775, 2002.

[12] B. Øksendal, *Stochastic differential equations, An introduction with applications*, 5th ed. Berlin Heidelberg New York: Springer-Verlag, 1998.

[13] H. J. Kushner, "On the stochastic maximum principle: Fixed time of control," *J. Math. Anal. Appl.*, vol. 11, pp. 78–92, 1965.

[14] S. Peng, "A general stochastic maximum principle for optimal control problems," *SIAM J. Contr. Optimiz.*, vol. 28, no. 4, pp. 966–979, 1990.

[15] J. M. Bismut, "An introductory approach to duality in optimal stochastic control," *SIAM Review*, vol. 20, pp. 62–78, 1978.

[16] N. E. Karoui and L. Mazliak, Eds., *Backward Stochastic Differential Equations*. Longman, 1997.

[17] S. Satoh, K. Fujimoto, and S. Hyon, "Biped gait generation via iterative learning control including discrete state transitions," in *Proc. 17th IFAC World Congress*, 2008, pp. 1729–1734.

[18] ——, "A framework for optimal gait generation via learning optimal control using virtual constraint," in *Proc. IEEE/RSJ Int. Conf. Intelligent Robots and Systems*, 2008, pp. 3426–3432.