

BOUNDED STABILIZATION OF A CLASS OF STOCHASTIC PORT-HAMILTONIAN SYSTEMS *

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Introduction. Since there possibly exist nonlinearity and uncertainty in controlling dynamical plants, the stabilization of nonlinear stochastic systems has been studied by many researchers. The literature [4] deals with a stochastic version of the control Lyapunov function approach. In [1], a stochastic output feedback stabilization controller based on the backstepping technique is proposed. A stabilization method based on stochastic passivity is introduced in [5]. In [12], the authors introduced stochastic port-Hamiltonian systems (SPHS's) as extension of deterministic port-Hamiltonian systems [9], and proposed a systematic stabilization method. This method is based on the stochastic passivity and the stochastic generalized canonical transformation [12], which is a pair of coordinate and feedback transformations preserving the SPHS structure, and is a stochastic version of the result in [6].

However, many stabilization methods including the above ones assume that the noise vanishes at the origin. Stochastic bounded stability, e.g., [7, 13, 8] is a useful concept for a system under persistent disturbances. The literature [2] introduced the concept of a noise-to-state stability, which is extension of the input-to-state stability property. In this paper, we consider bounded stabilization of a class of SPHS's based on the bounded stability concept, called (Q_0, Q_1, ρ) -stability [7]. We consider mechanical systems in the presence of noise, and derive conditions for the controller gain and design parameters under which the state remains bounded in probability. Moreover, the probability and the bound for the state can be assigned. In passivity-based control of a deterministic Hamiltonian system, an energy-based Lyapunov function is often used. Since, however, the time variation of the function depends only on a part of the state, boundedness of the state can not be guaranteed in the case of SPHS with noise which does not vanish at the origin. To solve this problem, we equip a stochastic Lyapunov function whose time variation involves all of the state variables.

Preliminaries. We consider a class of stochastic port-Hamiltonian systems (SPHS's) [12], which is described by the following Itô stochastic differential equation:

$$\begin{cases} \begin{pmatrix} dq \\ dp \end{pmatrix} = \begin{pmatrix} 0 & I_m \\ -I_m & -D_p \end{pmatrix} \begin{pmatrix} \frac{\partial H(q,p)}{\partial q} \\ \frac{\partial H(q,p)}{\partial p} \end{pmatrix} dt + \begin{pmatrix} 0 \\ G(q) \end{pmatrix} u dt + \begin{pmatrix} 0 \\ h_p(q,p) \end{pmatrix} dw \\ y = G(q)^\top \frac{\partial H(q,p)}{\partial p} = G(q)^\top M(q)^{-1} p \end{cases} \quad (0.1)$$

with the Hamiltonian $H(q, p) = \frac{1}{2}p^\top M(q)^{-1}p + U(q)$, where $q, p \in \mathbb{R}^m$, a symmetric positive definite matrix $M(q)$ denotes the inertia matrix, a function $U(q)$ denotes potential energy, and is assumed to be sufficiently differentiable positive definite. We

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have proposed a way to assign a proper potential energy to a SPHS in [12], and the literatures [6, 11] for the deterministic Hamiltonian systems are also useful. A positive semidefinite matrix $D_p \in \mathbb{R}^{m \times m}$ denotes the viscous friction coefficients. $u \in \mathbb{R}^l$ represents the control input, and $G(q) \in \mathbb{R}^{m \times l}$ is assumed to be a full-rank matrix for all q . $w(t) \in \mathbb{R}^r$ denotes a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is a sample space, \mathcal{F} is the sigma algebra of the observable random events and \mathcal{P} is a probability measure on Ω . \mathcal{F}_s represents the sigma algebra generated by $\{x(s) \mid 0 \leq s \leq t\}$, where $x := (q^\top, p^\top)^\top$. $h_p(q, p) \in \mathbb{R}^{m \times r}$ represents the noise port. The norm of a matrix A is defined as $\|A\| := \sqrt{\lambda_{\max}(A^\top A)}$, where $\lambda_{\max}(\cdot)$ represents the maximum eigenvalue. We assume that the Hamiltonian H is sufficiently differentiable, and that h_p satisfies the local Lipschitz condition and the linear growth condition, i.e, for all q, p , there exists a positive constant K_h such that $\|h_p(q, p)\|^2 \leq K_h(1 + \|q\|^2 + \|p\|^2)$. We define the region $\mathcal{D}(\delta_0, \delta_1)$ for $0 < \delta_0 < \delta_1$ as $\mathcal{D}(\delta_0, \delta_1) := \{x \mid \delta_0 \leq \|x\| \leq \delta_1\}$. We define the infinitesimal generator $\mathcal{L}(\cdot)$ to calculate the time variation of a stochastic Lyapunov function along x governed by Eq. (0.1).

DEFINITION 0.1. *Consider the following Itô type stochastic system as $dx = f(x) dt + g(x)u dt + h(x)dw$, with sufficiently differentiable functions f , g and h . Then, the infinitesimal generator for the stochastic process of the system is defined as $\mathcal{L}_u(\cdot) := \frac{\partial(\cdot)}{\partial x}(f + gu) + \frac{1}{2}\text{tr}\left\{\frac{\partial^2(\cdot)}{\partial x^2}hh^\top\right\}$. We can obtain the expectation of the time variation of a stochastic Lyapunov function $V(x)$ by calculating $\mathcal{L}_u(V)$ along a sample path x with an input u . Then, we introduce notion of (Q_0, Q_1, ρ) -stability [7] in order to consider the stochastic bounded stability.*

DEFINITION 0.2. [7] *The systems is (Q_0, Q_1, ρ) -stable if and only if for any initial condition $x(0) \in Q_0$, the probability with respect to the sample path $x(t)$ satisfies $\mathcal{P}\{x(t) \in Q_1, \text{ for } 0 \leq t < \infty\} \geq \rho$.*

Main results. We consider the following stochastic Lyapunov function $V(x)$ and the feedback input:

$$V(x) = \frac{a_1}{2}p^\top M(q)^{-1}p + a_2 \frac{\partial U(q)}{\partial q} M(q)^{-1}p + a_1 U(q), \quad u = -C(q, p)y, \quad (0.2)$$

where a_1 and a_2 are positive constants, and a positive semidefinite matrix $C(q, p) \in \mathbb{R}^{l \times l}$ represents the feedback gain. These design parameters should be chosen later. The closed-loop system of (0.1) with the feedback controller (0.2) is given by

$$\begin{pmatrix} dq \\ dp \end{pmatrix} = \begin{pmatrix} 0 & I_m \\ -I_m & -\bar{D}_p(q, p) \end{pmatrix} \begin{pmatrix} \frac{\partial H(q, p)}{\partial q} \\ \frac{\partial H(q, p)}{\partial p} \end{pmatrix} dt + \begin{pmatrix} 0 \\ h_p(q, p) \end{pmatrix} dw, \quad (0.3)$$

with the new dissipation matrix $\bar{D}_p(q, p) := D_p + G(q)C(q, p)G(q)^\top$. Now, we calculate the time variation of the Lyapunov function $V(x)$ along the closed-loop system (0.3). From Eq. (0.2), $\mathcal{L}_u(V)$ is obtained as

$$\begin{aligned} \mathcal{L}_u(V) &= \frac{\partial V}{\partial q} \frac{\partial H^\top}{\partial p} - \frac{\partial V}{\partial p} \left(\frac{\partial H^\top}{\partial q} + \bar{D}_p \frac{\partial H^\top}{\partial p} \right) + \frac{1}{2} \text{tr} \left\{ \frac{\partial^2 V}{\partial p^2} h_p h_p^\top \right\} \\ &= -p^\top M^{-1} \left(a_1 \bar{D}_p - a_2 \frac{\partial^2 U}{\partial q^2} \right) M^{-1} p - a_2 \frac{\partial U}{\partial q} M^{-1} \frac{\partial U^\top}{\partial q} + a_2 \frac{\partial U}{\partial q} N(q, p) p + \frac{1}{2} \text{tr} \left\{ \frac{\partial^2 V}{\partial p^2} h_p h_p^\top \right\}, \end{aligned} \quad (0.4)$$

where $N(q, p) := -\frac{1}{2}M^{-1} \frac{\partial M^{-1}p}{\partial q} + \frac{\partial M^{-1}p}{\partial q} M^{-1} - M^{-1} \bar{D}_p M^{-1}$. We can construct a symmetric matrix $\tilde{D}_p(q, p) := (\bar{D}_p + \bar{D}_p^\top)/2$ so that $p^\top M^{-1} \tilde{D}_p M^{-1} p = p^\top M^{-1} \bar{D}_p M^{-1} p$

for all q and p holds. By using $\tilde{D}_p(q, p)$, Eq. (0.4) is rewritten as

$$\begin{aligned}\mathcal{L}_u(V) &= \frac{1}{2} \operatorname{tr} \left\{ \frac{\partial^2 V}{\partial p^2} h_p h_p^\top \right\} - \left(\frac{\partial U}{\partial q}, p^\top \right) \begin{pmatrix} a_2 M^{-1} & \frac{a_2}{2} N^\top \\ \frac{a_2}{2} N & M^{-1} \left(a_1 \tilde{D}_p - a_2 \frac{\partial^2 U}{\partial q^2} \right) M^{-1} \end{pmatrix} \begin{pmatrix} \frac{\partial U}{\partial q} \\ p \end{pmatrix} \\ &=: \frac{1}{2} \operatorname{tr} \left\{ \frac{\partial^2 V}{\partial p^2} h_p h_p^\top \right\} - \left(\frac{\partial U}{\partial q}, p^\top \right) \Xi(x) \begin{pmatrix} \frac{\partial U}{\partial q} \\ p \end{pmatrix},\end{aligned}\quad (0.5)$$

where the matrix $\Xi(x) \in \mathbb{R}^{2m \times 2m}$ becomes symmetric from the definitions of \tilde{D}_p .

Before showing the main results, we prepare the following lemma.

LEMMA 0.3. *Consider the system of the form (0.1), the feedback input u in (0.2), and the region $\mathcal{D}(\delta_0, \delta_1)$ with some δ_0 and δ_1 . Suppose that there exist positive constants $M_m, K_{Um}, K_{Vm}(a_1, a_2), K_{VM}(a_1, a_2)$ and $K_{\Xi m}(a_1, a_2)$, which satisfy the following inequalities for $\forall x \in \mathcal{D}(\delta_0, \delta_1)$ with a_1 and a_2 in (0.2):*

$$M_m \leq \|M(q)\|, \quad K_{Um} \|q\|^2 \leq \left\| \frac{\partial U(q)}{\partial q} \right\|^2, \quad (0.6)$$

$$K_{Vm}(a_1, a_2) \|x\|^2 \leq V(x) \leq K_{VM}(a_1, a_2) \|x\|^2, \quad K_{\Xi m}(a_1, a_2) \leq \Xi(x). \quad (0.7)$$

Then, a sufficient condition under which $\mathcal{L}_u(V)$ with respect to V in (0.2) becomes strictly negative in the region $\mathcal{D}(\delta_0, \delta_1)$ is that there exist positive constants a_1 and a_2 and the gain matrix $C(q, p)$ in (0.2) (see, also \tilde{D}_p and \tilde{D}_p) such that, in $\mathcal{D}(\delta_0, \delta_1)$,

(i) the matrix $a_1 M^{-1} \tilde{D}_p M^{-1} - a_2 M^{-1} \frac{\partial^2 U}{\partial q^2} M^{-1} + \frac{a_2}{4} N M N$ becomes positive definite,

(iii) and the inequality $\min\{K_{Um}, 1\} K_{\Xi m}(a_1, a_2) \delta_0^2 > \frac{(1+\delta_1^2) a_1 r K_h}{2M_m}$ holds.

Proof. Firstly, the matrix $\Xi(x)$ in (0.5) should be positive definite in $\mathcal{D}(\delta_0, \delta_1)$. From the Schur complement, the condition (i) is given as the necessary and sufficient condition for that. Secondly, we evaluate $\mathcal{L}_u(V)$ in (0.5). From the linear growth condition, and the boundedness of $M(q)$ in Eq. (0.6), the first term in Eq. (0.5), which results from the noise effect in Itô calculus, is evaluated as

$$\frac{1}{2} \operatorname{tr} \left\{ \frac{\partial^2 V}{\partial p^2} h_p h_p^\top \right\} = \frac{a_1}{2} \sum_{i=1}^r \lambda_i (h_p^\top M^{-1} h_p) \leq \frac{a_1 r}{2M_m} K_h (1 + \|q\|^2 + \|p\|^2). \quad (0.8)$$

From Eqs. (0.6), (0.7) and (0.8), $\mathcal{L}_u(V)$ is evaluated as

$$\begin{aligned}\mathcal{L}_u(V) &\leq -K_{\Xi m} \left\| \begin{pmatrix} \frac{\partial U}{\partial q} \\ p \end{pmatrix} \right\|^2 + \frac{a_1 r}{2M_m} K_h (1 + \|q\|^2 + \|p\|^2) \\ &\leq -\min\{K_{Um}, 1\} K_{\Xi m} \delta_0^2 + \frac{a_1 r K_h}{2M_m} \delta_1^2 + \frac{a_1 r K_h}{2M_m}.\end{aligned}\quad (0.9)$$

The rest condition (ii) follows from Eq. (0.9) and that $\mathcal{L}_u(V) < 0$ in $\mathcal{D}(\delta_0, \delta_1)$. \square

THEOREM 0.4. *Consider the system of the form (0.1) and the feedback input u in (0.2). Suppose that the conditions in Lemma 0.3 hold.*

For any bounded region parameter $\delta_1 \in \mathbb{R}, \delta_1 > 0$ and any assigned probability $\rho \in \mathbb{R}, 0 < \rho < 1$, the initial region parameter $\bar{\delta}_0$ is assigned by $\bar{\delta}_0 = \sqrt{\frac{(1-\rho)K_{Vm}}{K_{VM}}} \delta_1$. Then, under the conditions (i) and (ii) in Lemma 0.3 with the bounded region parameter δ_1 and any initial region parameter $0 < \delta_0 < \bar{\delta}_0$, the system is (Q_0, Q_1, ρ) -stable (see, Definition 0.2), where Q_0 and Q_1 are given by

$$Q_0 = \{x \in \mathbb{R}^{2m} \mid x \in \mathcal{D}(\delta_0, \bar{\delta}_0)\}, \quad Q_1 = \{x \in \mathbb{R}^{2m} \mid \|x\| < \delta_1\}. \quad (0.10)$$

Further, the following probability inequality $\mathcal{P} \left\{ \sup_{0 \leq t < \infty} \|x(t)\| < \delta_1 \right\} > \rho$ is achieved.

Before proving the theorem, the stopped process [7] is introduced.

DEFINITION 0.5. [7] Define $t \cap s := \min\{t, s\}$. Suppose that $\tau_{\mathcal{D}}$ is the first time of exit of the process $x(s)$ from a set \mathcal{D} , i.e., $\tau_{\mathcal{D}} := \inf\{t \geq 0 \mid x(t) \notin \mathcal{D}\}$. Then, the stopped process $x(t \cap \tau_{\mathcal{D}})$ is defined as $x(t \cap \tau_{\mathcal{D}}) = x(t)$ when $t < \tau_{\mathcal{D}}$, and otherwise, $x(t \cap \tau_{\mathcal{D}}) = x(\tau_{\mathcal{D}})$.

Proof. For the proof, we define the following region with respect to the stochastic Lyapunov function:

$$\mathcal{D}^V(\lambda_0, \lambda_1) := \{x \in \mathbb{R}^{2m} \mid \lambda_0 < V(x) < \lambda_1\}. \quad (0.11)$$

From the assumption of the theorem and Lemma 0.3, $\mathcal{L}_u(V) < 0$ in $\mathcal{D}(\delta_0, \delta_1)$ holds. Then, it follows from Eqs. (0.7) and (0.11) that $\mathcal{L}_u(V) < 0$ in $\mathcal{D}^V(K_{VM}\delta_0^2, K_{Vm}\delta_1^2)$ holds. In what follows, we simply write \mathcal{D}^V as the region $\mathcal{D}^V(K_{VM}\delta_0^2, K_{Vm}\delta_1^2)$. Then, from the Dynkin's formula [3, 10], for any $0 \leq s \leq t$, we have

$$E[V(x(t \cap \tau_{\mathcal{D}^V}))] - E[V(x(s))] = E \left[\int_s^{t \cap \tau_{\mathcal{D}^V}} \mathcal{L}_u(V(x(\bar{t}))) d\bar{t} \right] < 0. \quad (0.12)$$

Since $E[V(x(t \cap \tau_{\mathcal{D}^V}))] < V(x(s))$ holds from Eq. (0.12), $\{V(x(t \cap \tau_{\mathcal{D}^V})); t \geq 0\}$ is a nonnegative supermartingale. It follows that, for any $t \geq 0$

$$E[V(x(t \cap \tau_{\mathcal{D}^V}))] < V(x(0)). \quad (0.13)$$

Since $V(x(t)) = K_{Vm}\delta_1^2$ for some t implies that the state x reaches the boundary of the region \mathcal{D}^V , we have

$$\mathcal{P} \left\{ \sup_{0 \leq t \leq \bar{t}} V(x(t)) \geq K_{Vm}\delta_1^2 \right\} = \mathcal{P} \left\{ V(x(\bar{t} \cap \tau_{\mathcal{D}^V})) = K_{Vm}\delta_1^2 \right\}, \quad \forall \bar{t} \geq 0. \quad (0.14)$$

According to the definition of the expectation of the positive definite function $V(x)$, the following inequality holds:

$$E[V(x(\bar{t} \cap \tau_{\mathcal{D}^V}))] \geq K_{Vm}\delta_1^2 \mathcal{P} \left\{ V(x(\bar{t} \cap \tau_{\mathcal{D}^V})) = K_{Vm}\delta_1^2 \right\}, \quad \forall \bar{t} \geq 0. \quad (0.15)$$

From Eqs. (0.13), (0.14), (0.15), and by taking limit as $\bar{t} \rightarrow \infty$, we obtain

$$\frac{V(x(0))}{K_{Vm}\delta_1^2} > \mathcal{P} \left\{ \sup_{0 \leq t < \infty} V(x(t)) \geq K_{Vm}\delta_1^2 \right\}. \quad (0.16)$$

It follows from Eq. (0.7) that if $\|x(0)\| \leq \bar{\delta}_0$, then $V(x(0)) \leq (1 - \rho)K_{Vm}\delta_1^2$ holds. Therefore, if $x(0)$ is chosen from Q_0 in (0.10), we have

$$\mathcal{P} \left\{ \sup_{0 \leq t < \infty} V(x(t)) < K_{Vm}\delta_1^2 \right\} > 1 - \frac{V(x(0))}{K_{Vm}\delta_1^2} \geq 1 - \frac{(1 - \rho)K_{Vm}\delta_1^2}{K_{Vm}\delta_1^2} = \rho. \quad (0.17)$$

Since $V(x(t)) < K_{Vm}\delta_1^2$ is a sufficient condition for $\|x(t)\| < \delta_1$ from Eq. (0.7), Eq. (0.17) implies that the asserted probability inequality holds. \square

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