A study of stochastic input-to-state stability of a class of stochastic port-Hamiltonian systems

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***Abstract*— The main purpose of this paper is to investigate the stochastic input-to-state stability (SISS) property of a class of stochastic port-Hamiltonian systems. We provide sufficient conditions for the system to achieve the SISS property.**

***Index Terms*— Stochastic input-to-state stability, Stochastic Hamiltonian systems, Nonlinear stochastic control.**
***AMS subject classifications*—- 60H10**

I. Introduction

Stochastic input-to-state stability (SISS) is well recognized as an important concept for analysis and synthesis of nonlinear stochastic systems. The notion of SISS is an extension of the deterministic input-to-state stability (ISS) concept [1], and has been improved by several researchers, e.g., [2], [3], [4]. Besides, as a practically important class of nonlinear stochastic systems, one of the authors has introduced stochastic port-Hamiltonian systems (SPHSs) in [5]. SPHS is an extension of the deterministic port-Hamiltonian system (PHS) [6], and it represents electrical and mechanical systems, electromechanical systems and systems with nonholonomic constraints with uncertainties such as system and measurement noises and modeling error.

The main purpose of this paper is to investigate the SISS property of a class of SPHSs. As mentioned in [7], there are few studies on robustness of PHSs against external disturbances, and this important subject has not been clarified for SPHSs. The key techniques in this paper are coordinate transformations, integral actions and feedback compensators derived from the stochastic generalized canonical transformations (SGCTs). Although we use particular coordinate transformations in [8], [7] and add an integral action in [7], there are two main differences amoung them. First, we revealed in [5] that an extra condition due to the stochastic noise is necessary to preserve the SPHS structure under a coordinate transformation compared to the case of PHS. Hence, the techniques in [8], [7] are not directly applied to SPHSs. SGCTs proposed in [5] provide conditions for the coordinate and feedback transformations preserving the SPHS structure, and we newly equip some feedback compensators based on SGCTs to solve this problem. Second, although the method in [7] which we use here only deals with matched deterministic disturbances, the proposed method can deal with matched and unmatched stochastic noises as well as the matched deterministic disturbances. Thanks to stochastic calculus, we explicitly provide sufficient conditions of the noise port of the plant system as well as some design parameters of the controller for achieving SISS property.

II. Preliminaries

We consider a class of stochastic port-Hamiltonian systems (SPHSs) [5], which is described by the following Itô stochastic differential equation:

$$\begin{pmatrix} \mathrm{d}q \\ \mathrm{d}p \end{pmatrix} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H(q,p)}{\partial q}^\top \\ \frac{\partial H(q,p)}{\partial p}^\top \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0 \\ I_m \end{pmatrix} u \, \mathrm{d}t + \begin{pmatrix} 0 \\ d_2 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} h_{11}(q,p) & 0 \\ 0 & h_{22}(q,p) \end{pmatrix} \begin{pmatrix} \mathrm{d}w_1 \\ \mathrm{d}w_2 \end{pmatrix} \quad (1)$$

with the Hamiltonian $H(q,p) = \frac{1}{2}p^\top M(q)^{-1}p + U(q)$, where $q, p \in \mathbb{R}^m$, a symmetric positive definite matrix $M(q)$ denotes the inertia matrix, a scalar function $U(q)$ denotes a potential energy, and is assumed to be a sufficiently differentiable positive definite function. We have proposed a way to assign a proper potential energy to a SPHS in [5]. Let $T(q) \in \mathbb{R}^{m\times m}$ be the square root of the matrix $M(q)^{-1}$, that is, $M(q)^{-1} = T(q)^2$ holds. $u \in \mathbb{R}^m$ represents the control input and $d_2 \in \mathbb{R}^m$ represents an essentially bounded measurable disturbance. d_2 is possibly time-varying. $w_1, w_2 \in \mathbb{R}^{m_w}$ denote standard Wiener processes defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is a sample space, $\mathcal{F}$ is the sigma algebra of the observable random events and $\mathcal{P}$ is a probability measure on Ω. $h_{11}(q,p), h_{22}(q,p) \in \mathbb{R}^{m\times m_w}$ represent the noise ports. In the sequel, we define the norm of a matrix A as $\|A\| := \sqrt{\lambda_{\max}(A^\top A)}$. Here, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ represent the minimum and maximum eigenvalues of $(\cdot)$, respectively. We use the Euclidean norm for vectors.

In order to calculate the expectation of the time variation of a Hamiltonian, we define the infinitesimal operator $\mathcal{L}(\cdot)$.

Definition 1: Consider the nonlinear stochastic system written in the sense of Itô:

$$\mathrm{d}x = f(x,u,d)\,\mathrm{d}t + h(x)\,\mathrm{d}w, \quad (2)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ $d \in \mathbb{R}^l$ represent the state, the control input and the disturbance, respectively. According to [9], it is assumed that f and h satisfy the linear growth condition and local Lipschitz condition with respect to their arguments for existence and uniqueness of a global solution to the system (2).

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Then, the infinitesimal generator for the stochastic process of the system (2) is defined as

$$\mathcal{L}(\cdot) := \frac{\partial(\cdot)}{\partial x} f + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial^2(\cdot)}{\partial x^2} hh^\top\right\}. \quad (3)$$

We equip the definitions of stochastic input-to-state stability (SISS) and SISS Lyapunov function based on [3]. Since the definition of SISS in [3] does not consider the plant system with a control input, we slightly modify the difinition.

Definition 2: Consider the system (2) with a fixed control input $u = v(x,t)$. Then, the closed-loop system is stochastic input-to-state stable if $\forall \epsilon > 0$, there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that

$$\mathcal{P}\left\{\|x(t)\| < \beta(\|x(0)\|, t) + \gamma\left(\sup_{0\le s\le t}\|d(s)\|\right)\right\} \ge 1 - \epsilon, \quad \forall t \ge 0\ , \quad \forall x(0) \in \mathbb{R}^n.$$

Proposition 1: [3] Consider the system (2) with a fixed control input $u = v(x,t)$, and suppose that there exist a C^2 function $V(x)$, class $\mathcal{K}_\infty$ functions α_1, α_2 and ρ, and a class $\mathcal{K}$ function α_3, such that

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|) \quad (4)$$

and with $\|x\| \ge \rho(\|d\|)$,

$$\mathcal{L}V \le -\alpha_3(\|x\|). \quad (5)$$

Then, the closed-loop system is SISS in Definition 2 .

Although the concept of SISS is extended to the case where the disturbance is assumed to be a state and time dependent stochastic process in [4], we do not consider the state dependent disturbance in this paper.

III. Main results

Before applying a main controller for SISS, we convert the system (1) into another one, where the inertia matrix is removed from its Hamiltonian. For this, we extend the coordinate transformation in [8] to a stochastic port-Hamiltonian system by using a SGCT in [5].

Proposition 2: Consider the system (1). The following coordinate transformation

$$\begin{pmatrix}\bar{q}\\ \bar{p}\end{pmatrix} = \begin{pmatrix} q \\ T(q)p\end{pmatrix}$$

and feedback input

$$u = -T(q)^{-1}\begin{pmatrix}\sum_{j,k=1}^m \frac{1}{2}\frac{\partial^2 [T]_{1,j}}{\partial q_k^2} p_j [h_{11}]_{k,k}^2 \\ \vdots \\ \sum_{j,k=1}^m \frac{1}{2}\frac{\partial^2 [T]_{m,j}}{\partial q_k^2} p_j [h_{11}]_{k,k}^2\end{pmatrix} + \bar{u},$$

where for a vector x, x_j denotes the j th element of x and for a matrix A, $[A]_{j,k}$ denotes the (j,k) th element of A, respectively, and $\bar{u}$ denotes a new input, convert the system (1) into the following another SPHS:

$$\begin{pmatrix}\mathrm{d}\bar{q}\\ \mathrm{d}\bar{p}\end{pmatrix} = \begin{pmatrix} 0 & T(\bar{q}) \\ -T(\bar{q}) & J_2(\bar{q},\bar{p})\end{pmatrix}\begin{pmatrix}\frac{\partial \bar{H}(\bar{q},\bar{p})}{\partial \bar{q}}^\top \\ \frac{\partial \bar{H}(\bar{q},\bar{p})}{\partial \bar{p}}^\top\end{pmatrix}\mathrm{d}t + \begin{pmatrix}0\\ I_m\end{pmatrix} v\,\mathrm{d}t + \begin{pmatrix}0 \\ T(\bar{q})d_2\end{pmatrix}\mathrm{d}t + \begin{pmatrix}\bar{h}_{11}(\bar{q},\bar{p}) & 0 \\ \bar{h}_{21}(\bar{q},\bar{p}) & \bar{h}_{22}(\bar{q},\bar{p})\end{pmatrix}\begin{pmatrix}\mathrm{d}w_1\\ \mathrm{d}w_2\end{pmatrix}. \quad (6)$$

Here, $v := T\bar{u}$ and

$$J_2(\bar{q},\bar{p}) := \left.\frac{\partial(Tp)}{\partial q}T - T\frac{\partial(Tp)}{\partial q}^\top\right|_{\substack{q=\bar{q}\\ p=T^{-1}\bar{p}}}$$

$$\bar{h}_{11}(\bar{q},\bar{p}) := \left. h_{11}(q,p)\right|_{\substack{q=\bar{q}\\ p=T^{-1}\bar{p}}}$$

$$\bar{h}_{21}(\bar{q},\bar{p}) := \left.\frac{\partial(Tp)}{\partial q}h_{11}(q,p)\right|_{\substack{q=\bar{q}\\ p=T^{-1}\bar{p}}}$$

$$\bar{h}_{22}(\bar{q},\bar{p}) := \left. Th_{22}(q,p)\right|_{\substack{q=\bar{q}\\ p=T^{-1}\bar{p}}}.$$

Proof: The assertion directly follows from Theorem 1 in [5]. ■

In order to endow the system with SISS property, we first add an integrator dynamics to the system. Then, coordinate transformation and feedback controller are equipped to add the proper damping and to preserve the SPHS structure. The integrator dynamics and coordinate transformation are the same as proposed in [7]. However, the feedback controller proposed here is different.

Theorem 1: Consider the system (6), and the following integrator dynamics

$$\mathrm{d}\bar{r} = (T + R_3)\frac{\partial U}{\partial \bar{q}} + R_3\bar{p} \quad (7)$$

and feedback controller

$$v = -\left(\frac{\partial^2 U}{\partial \bar{q}^2}T + J_2 + R_2 + R_3\right)\bar{p} - (R_2+R_3)\bar{r} - (T + R_2 + R_3)\frac{\partial U}{\partial \bar{q}} - \begin{pmatrix}\sum_{j=1}^m \frac{1}{2}\frac{\partial^3 U}{\partial \bar{q}_j^2 \partial \bar{q}_1}[\bar{h}_{11}]_{j,j}^2 \\ \vdots \\ \sum_{j=1}^m \frac{1}{2}\frac{\partial^3 U}{\partial \bar{q}_j^2 \partial \bar{q}_m}[\bar{h}_{11}]_{j,j}^2\end{pmatrix}, \quad (8)$$

where $R_2, R_3 \in \mathbb{R}^{m\times m}$ denote symmetric positive definite matrices, respectively. Then, the closed-loop system described in the new coordinate

$$z := \begin{pmatrix} z_1\\ z_2\\ z_3\end{pmatrix} = \begin{pmatrix}\bar{q} \\ \bar{p} + \frac{\partial U}{\partial \bar{q}} + \bar{r} \\ \bar{r}\end{pmatrix} =: \Psi(\bar{q},\bar{p},\bar{r}) \quad (9)$$

is converted into another SPHS with the Hamiltonian

$$H_z(z) = U(z_1) + \frac{1}{2}z_2^\top z_2 + \frac{1}{2}z_3^\top z_3.$$

Moreover, suppose that for all $q, p \in \mathbb{R}^m$, there exist positive constants such that

$$k_T I_m \le T(q) \le K_T I_m \tag{10}$$

$$k_{U1}\|q\| \le \left\| \frac{\partial U(q)}{\partial q}^\top \right\| \tag{11}$$

$$k_{U2} I_m \le \frac{\partial^2 U(q)}{\partial q^2} \le K_{U2} I_m \tag{12}$$

$$\|h_{11}(q,p)\| \le K_{h11} \left\| \begin{pmatrix} q \\ p \end{pmatrix} \right\| \tag{13}$$

$$\left\| \frac{\partial (T(q)p)}{\partial q} h_{11}(q,p) \right\| \le K_{h21} \left\| \begin{pmatrix} q \\ p \end{pmatrix} \right\| \tag{14}$$

$$\|h_{22}(q,p)\| \le K_{h22} \left\| \begin{pmatrix} q \\ p \end{pmatrix} \right\|. \tag{15}$$

Then, if the following condition holds:

$$k_T > \frac{9m^2 K_{U2}(k_{U1}^2+1)}{2k_{U1}^2} \left((1+K_{\nabla^2 U})K_{h11}+K_{h21}+K_T K_{h22}\right)^2 =: K_z, \tag{16}$$

the closed loop system in the coordinate (9) with R_2 and R_3 such that

$$k_{R_2} := \lambda_{min}(R_2) > K_z + \frac{K_T}{2} \tag{17}$$

$$k_{R_3} := \lambda_{min}(R_3) > K_z \tag{18}$$

becomes SISS with respect to d_2 with SISS Lyapunov function $H_z(z)$.

Proof: The closed-loop system with the integrator dynamics (7) and the feedback controller (8) is given by

$$\begin{pmatrix} \mathrm{d}\bar{q} \\ \mathrm{d}\bar{p} \\ \mathrm{d}\bar{r} \end{pmatrix} = \begin{pmatrix} 0 \\ T d_2 \\ 0 \end{pmatrix} \mathrm{d}t - \begin{pmatrix} 0 \\ I_m \\ 0 \end{pmatrix} \begin{pmatrix} \Sigma_{j=1}^m \frac{1}{2} \frac{\partial^3 U}{\partial \bar{q}_j^2 \partial \bar{q}_1} [\bar{h}_{11}]_{j,j}^2 \\ \vdots \\ \Sigma_{j=1}^m \frac{1}{2} \frac{\partial^3 U}{\partial \bar{q}_j^2 \partial \bar{q}_m} [\bar{h}_{11}]_{j,j}^2 \end{pmatrix} \mathrm{d}t +$$
$$\begin{pmatrix} T\bar{p} \\ -(2T+R_2+R_3)\frac{\partial U}{\partial \bar{q}} - \left(\frac{\partial^2 U}{\partial \bar{q}^2} T + R_2 + R_3\right)\bar{p} - (R_2+R_3)\bar{r} \\ (T+R_3)\frac{\partial U}{\partial \bar{q}} + R_3 \bar{p} \end{pmatrix} \mathrm{d}t$$
$$+ \begin{pmatrix} \bar{h}_{11}(\bar{q},\bar{p}) & 0 \\ \bar{h}_{21}(\bar{q},\bar{p}) & \bar{h}_{22}(\bar{q},\bar{p}) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}w_1 \\ \mathrm{d}w_2 \end{pmatrix}. \tag{19}$$

Then, the system (19) can be described in the new coordinate z in (9) as

$$\begin{pmatrix} \mathrm{d}z_1 \\ \mathrm{d}z_2 \\ \mathrm{d}z_3 \end{pmatrix} = \begin{pmatrix} -T & T & -T \\ -T & -R_2 & -R_3 \\ T & R_3 & -R_3 \end{pmatrix} \frac{\partial H_z(z)}{\partial z}^\top \mathrm{d}t$$
$$+ \begin{pmatrix} 0 \\ T d_2 \\ 0 \end{pmatrix} \mathrm{d}t + \begin{pmatrix} h_{z11}(z) & 0 \\ h_{z21}(z) & h_{z22}(z) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}w_1 \\ \mathrm{d}w_2 \end{pmatrix}, \tag{20}$$

where

$$h_{z11}(z) := \bar{h}_{11}(\bar{q},\bar{p}) \Big|_{\substack{\bar{q}=z_1 \\ \bar{p}=z_2-\frac{\partial U}{\partial z_1}-z_3}}$$

$$h_{z21}(z) := \bar{h}_{21}(\bar{q},\bar{p}) + \frac{\partial^2 U}{\partial \bar{q}^2}\bar{h}_{11}(\bar{q},\bar{p}) \Big|_{\substack{\bar{q}=z_1 \\ \bar{p}=z_2-\frac{\partial U}{\partial z_1}-z_3}}$$

$$h_{z22}(z) := \bar{h}_{22}(\bar{q},\bar{p}) \Big|_{\substack{\bar{q}=z_1 \\ \bar{p}=z_2-\frac{\partial U}{\partial z_1}-z_3}}.$$

Note that we use the fact that

$$\frac{1}{2}\begin{pmatrix} \mathrm{tr}\left\{\frac{\partial^2 \Psi_1}{\partial(\bar{q},\bar{p},\bar{r})} h_z h_z^\top\right\} \\ \vdots \\ \mathrm{tr}\left\{\frac{\partial^2 \Psi_{3m}}{\partial(\bar{q},\bar{p},\bar{r})} h_z h_z^\top\right\} \end{pmatrix} = \begin{pmatrix} 0 \\ I_m \\ 0 \end{pmatrix} \begin{pmatrix} \Sigma_{j=1}^m \frac{1}{2} \frac{\partial^3 U}{\partial \bar{q}_j^2 \partial \bar{q}_1} [\bar{h}_{11}]_{j,j}^2 \\ \vdots \\ \Sigma_{j=1}^m \frac{1}{2} \frac{\partial^3 U}{\partial \bar{q}_j^2 \partial \bar{q}_m} [\bar{h}_{11}]_{j,j}^2 \end{pmatrix}$$

in deriving Eq. (20), where

$$h_z(z) := \begin{pmatrix} h_{z11}(z) & 0 \\ h_{z21}(z) & h_{z22}(z) \\ 0 & 0 \end{pmatrix}.$$

Eq. (20) proves the first assertion.

Now, we calculate $\mathcal{L}H_z$ along the system (20) as

$$\mathcal{L}H_z = -\frac{\partial H_z}{\partial z}\,\mathrm{diag}\{T, R_2, R_3\}\frac{\partial H_z}{\partial z}^\top + \frac{\partial H_z}{\partial z_2} T d_2 + \frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial^2 H_z}{\partial z^2} h_z h_z^\top\right\}. \tag{21}$$

In order to further investigate $\mathcal{L}H_z$, we evaluate the last term in Eq. (21). From the conditions (10), (12), (13), (14) and (15), we have

$$\begin{aligned} \|h_z(z)\| &\le \sqrt{3m}\left(\|h_{z11}(z)\| + \|h_{z21}(z)\| + \|h_{z22}(z)\|\right) \\ &\le \sqrt{3m}\left((1+K_{\nabla^2 U})K_{h11} + K_{h21} + K_T K_{h22}\right) \\ &\quad \times \left\| \begin{pmatrix} z_1 \\ z_2 - \frac{\partial U}{\partial z_1}^\top - z_3 \end{pmatrix} \right\|. \end{aligned} \tag{22}$$

It follows from the condition (11) that

$$\left\| \begin{pmatrix} z_1 \\ z_2 - \frac{\partial U}{\partial z_1}^\top - z_3 \end{pmatrix} \right\|^2 \le \left(\frac{1}{k_{U1}^2}+1\right)\left(\left\| \frac{\partial U}{\partial z_1}^\top \right\|^2 + \|z_2\|^2 + \|z_3\|^2 \right). \tag{23}$$

Then, Eqs. (22), (23) yield

$$\|h_z(z)\|^2 \le 3m\left((1+K_{\nabla^2 U})K_{h11} + K_{h21} + K_T K_{h22}\right)^2 \times \left(\frac{1}{k_{U1}^2}+1\right)\left(\left\| \frac{\partial U}{\partial z_1}^\top \right\|^2 + \|z_2\|^2 + \|z_3\|^2 \right). \tag{24}$$

By using Eq. (24), the last term in Eq. (21) can be evaluated as

$$\begin{aligned}\frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial^2 H_z}{\partial z^2}h_z h_z^\top\right\} &\leq \frac{3m}{2}\lambda_{max}\left\{\frac{\partial^2 H_z}{\partial z^2}h_z h_z^\top\right\}\\ &\leq \frac{3mK_{U2}}{2}\|h_z(z)\|^2\\ &\leq K_z\left(\left\|\frac{\partial U}{\partial z_1}^\top\right\|^2+\|z_2\|^2+\|z_3\|^2\right), \end{aligned}\tag{25}$$

where K_z is given in Eq. (16). Therefore, from Eqs. (21) and (25), we have

$$\begin{aligned}\mathcal{L}H_z \leq &-(k_T-K_z)\left\|\frac{\partial U}{\partial z_1}^\top\right\|^2-(k_{R_2}-K_z)\|z_2\|^2\\ &-(k_{R_3}-K_z)\|z_3\|^2+K_T z_2^\top d_2\\ \leq &-k_{U1}^2(k_T-K_z)\|z_1\|^2-\left(k_{R_2}-K_z-\frac{K_T}{2}\right)\|z_2\|^2\\ &-(k_{R_3}-K_z)\|z_3\|^2+\frac{K_T}{2}\|d_2\|^2. \end{aligned}\tag{26}$$

It follows from the conditions (16), (17) and (18), and Proposition 1 that the closed loop system in the coordinate z is SISS with respect to d_2 with SISS Lyapunov function $H_z(z)$. This proves the theorem. ■

IV. CONCLUSION

This paper has investigated a SISS property of a class of SPHSs. By using particular coordinate transformations and integral action, and feedback compensators derived from SGCTs, we have clarified robustness against matched deterministic disturbances, and matched and unmatched stochastic noises for SPHSs.

REFERENCES

[1] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Autom. Contr.*, vol. 34, no. 4, pp. 435–443, 1989.

[2] J. Tsinias, "Stochastic input-to-state stability and applications to global feedback stabilization," *Int. J. Control*, vol. 71, no. 5, pp. 907–930, 1998.

[3] C. Tang and T. Başar, "Stochastic stability of singularly perturbed nonlinear systems," in *Proc. 40th IEEE Conf. on Decision and Control*, 2001, pp. 399–404.

[4] S.-J. Liu, J.-F. Zhang, and Z.-P. Jiang, "A notion of stochastic input-to-state stability and its application to stability of cascaded stochastic nonlinear systems," *Acta Mathematicae Applicatae Sinica*, vol. 24, no. 1, pp. 141–156, 2008.

[5] S. Satoh and K. Fujimoto, "Passivity based control of stochastic port-Hamiltonian systems," *IEEE Trans. Autom. Contr.*, vol. 58, no. 5, pp. 1139–1153, 2013.

[6] B. Maschke and A. J. van der Schaft, "Port-controlled Hamiltonian systems: modelling origins and system theoretic properties," in *Proc. 2nd IFAC Symp. Nonlinear Control Systems*, 1992, pp. 282–288.

[7] J. G. Romero, A. Donaire, and R. Ortega, "Robust energy shaping control of mechanical systems," *Systems & Control Letters*, vol. 62, no. 9, pp. 770–780, 2013.

[8] A. Venkatraman, R. Ortega, I. Sarras, and A. J. van der Schaft, "Speed observation and position feedback stabilization of partially linearizable mechanical systems," *IEEE Trans. Autom. Contr.*, vol. 55, no. 5, pp. 1059–1074, 2010.

[9] L. Arnold, *Stochastic Differential Equations: Theory and Applications*. New York: Wiley, 1973.