Stabilization of Time-varying Stochastic Port-Hamiltonian Systems Based on Stochastic Passivity

S. Satoh * **K. Fujimoto** **

* *Hiroshima University, 1-4-1, Kagamiyama, Higashi-Hiroshima, 739-8527, Japan (e-mail:* satoh@hiroshima-u.ac.jp*).*
** *Nagoya University, Furo-cho, Chikusa-ku, Nagoya, 464-8603, Japan (e-mail:* fujimoto@nagoya-u.jp*).*

Abstract: The authors have introduced stochastic port-Hamiltonian systems and have clarified some of their properties. Stochastic port-Hamiltonian systems are extension of deterministic port-Hamiltonian systems, which are used to express various deterministic passive systems. However, since only time-invariant case has been considered in our previous results, the aim of this paper is to extend them to time-varying case. Finally, we propose a stabilization method based on passivity and the stochastic generalized canonical transformation, which is a pair of coordinate and feedback transformations preserving the stochastic Hamiltonian structure.

Keywords: stochastic Hamiltonian systems, passive stochastic systems, stochastic stability, nonlinear stochastic control, time-varying systems

1. INTRODUCTION

There exist various disturbances such as measurement noise, modeling error and so on in controlling real plants. Since they sometimes cause performance degradation or destabilization of the plant system, it is important to consider them. Stochastic control theory is one of the efficient ways which can take such disturbances into account. Theories and techniques for the deterministic dynamical systems described by ordinary differential equations are applied to stochastic ones described by stochastic differential equations (Ikeda and Watanabe [1989]). Lyapunov function approaches to the stochastic stability of stochastic systems are introduced in (Bucy [1965], Kushner [1967]). In these frameworks, nonnegative supermartingales are used as stochastic Lyapunov functions, and asymptotic convergence of sample paths is proven by the martingale convergence theorem (see also (Has'minskii [1980], Mao [1991, 1999])). The notion of stochastic passivity for the stochastic systems is introduced in (Florchinger [1999]). One can utilize the well-known results of the stabilization method for the deterministic passive systems (Byrnes et al. [1991]) to achieve the asymptotic stability for the stochastic nonlinear systems in probability.

The authors have introduced stochastic port-Hamiltonian systems and have clarified some of their properties in (Satoh and Fujimoto [2008a]). Stochastic port-Hamiltonian systems are extension of deterministic ones (Maschke and van der Schaft [1992], van der Schaft [1996]), which are used to express various deterministic passive systems. It has not been considered so far a non-autonomous stochastic Hamiltonian system whose dynamics is described by a stochastic differential equation written in the sense of Itô. We revealed that some properties such as invariance under a class of transformations and passivity do not generally hold for stochastic port-Hamiltonian systems despite the case of the deterministic port-Hamiltonian ones. However, since only time-invariant case is considered in our previous results, the aim of this paper is to extend them to time-varying case. Some motivations for considering the time-varying case are as follows. One is that trajectory tracking control problem reduces to stabilization problem of an error dynamics, which is generally a time-varying system. The other is that systems with nonholonomic constraints can not be asymptotically stabilized around a fixed point under any continuous static feedback controller (Brockett [1983]). A time-varying feedback controller is one of the solutions of the above problem (Pomet [1992]).

In this paper, firstly, we will derive a condition under which the time-varying stochastic port-Hamiltonian system maintains stochastic passivity. Secondly, we will extend the stochastic generalized canonical transformations, which are pairs of coordinate and feedback transformations preserving the stochastic Hamiltonian structure, to the time-varying case. The original idea of such transformations was proposed in (Fujimoto and Sugie [2001]) for the deterministic port-Hamiltonian systems. Thirdly, we will provide a condition that the transformed system by this transformation becomes stochastic passive, and will propose a stabilization method based on passivity and the stochastic generalized canonical transformation. Finally, numerical simulations demonstrate the effectiveness of the proposed method.

2. TIME-VARYING STOCHASTIC PORT-HAMILTONIAN SYSTEMS AND THEIR PASSIVITY

We have considered time-invariant stochastic port -Hamiltonian systems whose dynamics are described by stochastic differential equations written in the sense of Itô (Satoh and Fujimoto [2008a]). In this paper, we extend the form of these systems into the following time-varying one:

$$\begin{cases} \mathrm{d}x = (J(x,t) - R(x,t))\dfrac{\partial H(x,t)}{\partial x}^{\top}\mathrm{d}t + g(x,t)u\,\mathrm{d}t \\ \qquad +h(x,t)\,\mathrm{d}w \\ y = g(x,t)^{\top}\dfrac{\partial H(x,t)}{\partial x}^{\top} \end{cases}. \quad (1)$$

Here $x(t) \in \mathbb{R}^n, u(t), y(t) \in \mathbb{R}^m$ describe the state, the input and the output, respectively. The structure matrix $J(x,t) \in \mathbb{R}^{n\times n}$ and the dissipation matrix $R(x,t) \in \mathbb{R}^{n\times n}$ are skew-symmetric and symmetric positive semi-definite for all x and t, respectively. $g(x,t) \in \mathbb{R}^{n\times m}$ represents the control port. $w(t) \in \mathbb{R}^r$ is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is a sample space, $\mathcal{F}$ is the sigma algebra of the observable random events and $\mathcal{P}$ is a probability measure on Ω. $h(x,t) \in \mathbb{R}^{n\times r}$ represents the noise port. We suppose that a Hamiltonian $H(x,t)$ is a sufficiently differentiable function and $h(0,t) = 0$. Moreover, $g(x,t)$ and $h(x,t)$ satisfy reasonable sufficient conditions for the local existence and uniqueness of the solutions. For the details on such conditions, see (Ikeda and Watanabe [1989], Øksendal [1998]). We call stochastic systems described by the form (1) **time-varying stochastic port-Hamiltonian systems**.

Remark 1. Suppose that the noise port does not exist, that is $h(x,t) \equiv 0$ for any x and t, then the system (1) reduces to the conventional deterministic port-Hamiltonian system (Maschke and van der Schaft [1992], van der Schaft [1996]) as a special case.

In (Fujimoto and Sugie [2001]), a passivity condition was given, under which a time-varying deterministic port-Hamiltonian system becomes passive, and the generalized canonical transformations were proposed, which are pairs of coordinate and feedback transformations preserving the port-Hamiltonian structure. In this section and the next, we extend these results for the deterministic case to a time-varying port-Hamiltonian system (1).

In this section, firstly, we investigate the **stochastic passivity** introduced by (Florchinger [1999]) of the system (1). However, since the literature (Florchinger [1999]) deals with the property for only time-invariant stochastic systems, let us extend this notion to time-varying stochastic systems in a manner analogous to the deterministic time-varying case (Sastry [1999], Fujimoto and Sugie [2001]).

Definition 1. Consider the followingstochastic system

$$\begin{cases} \mathrm{d}x = f(x,u,t)\,\mathrm{d}t + h(x,t)\,\mathrm{d}w \\ y = s(x,u,t) \end{cases}, \quad (2)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n, h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times r}$ and $s : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ are sufficiently differentiable functions and they satisfy $f(0,0,t) = 0$, $h(0,t) = 0$ and $s(0,0,t) = 0$, respectively. Then the system (2) is said to be **stochastic passive** if there exists a non-negative function $V(x,t)$: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that $V(x,t) \geq V(0,t) = 0$ and it satisfies $\mathcal{L}V(x,t) \leq s(x,u,t)^{\top}u$. Here $\mathcal{L}(\cdot)$ represents the following infinitesimal generator for time-varying functions:

$$\mathcal{L}(\cdot) := \frac{\partial(\cdot)}{\partial t} + \frac{\partial(\cdot)}{\partial x}f + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial(\cdot)}{\partial x}\right)^{\top} hh^{\top}\right\}. \quad (3)$$

The following lemma characterizes stochastic passivity of time-varying stochastic port-Hamiltonian systems.

Lemma 1. Consider the time-varying stochastic port-Hamiltonian system (1). Suppose that a Hamiltonian $H(x,t)$ is a non-negative function such that $H(x,t) \geq H(0,t) = 0$ Then, the system is stochastic passive if and only if the following inequality holds:

$$\frac{\partial H(x,t)}{\partial t} + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial H(x,t)}{\partial x}\right)^{\top} h(x,t)h(x,t)^{\top}\right\} \leq \frac{\partial H(x,t)}{\partial x}R(x,t)\frac{\partial H(x,t)}{\partial x}^{\top}. \quad (4)$$

Proof Firstly, the necessity of the condition (4) is shown. By utilizing Eq. (3), $\mathcal{L}H(x,t)$ is calculated as

$$\begin{aligned} \mathcal{L}H(x,t) &= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\left((J-R)\frac{\partial H}{\partial x}^{\top} + gu\right) \\ &\quad + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\right)^{\top} hh^{\top}\right\} \\ &= \frac{\partial H}{\partial t} - \frac{\partial H}{\partial x}R\frac{\partial H}{\partial x}^{\top} + y^{\top}u + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\right)^{\top} hh^{\top}\right\}. \quad (5) \end{aligned}$$

The last equality follows from that

$$\frac{\partial H(x,t)}{\partial x}J(x,t)\frac{\partial H(x,t)}{\partial x}^{\top} = 0$$

with skew-symmetric matrix $J(x,t)$ in (1). Suppose that the system (1) is stochastic passive. Then the definition of stochastic passivity and (5) prove the necessity.

Secondly, the sufficiency of the condition (4) is shown. Substituting the inequality (4) for Eq. (5), we obtain $\mathcal{L}H(x,t) \leq y^{\top}u$. According to the definition of stochastic passivity, this proves the sufficiency of Eq. (4).

Remark 2. Consider that we apply Lemma 1 to the deterministic port-Hamiltonian system with the Hamiltonian $H(x,t)$ satisfying $H(x,t) \geq H(0,t) = 0$. For the system, $h(x,t) \equiv 0$ holds. Then the condition (4) reduces to

$$\frac{\partial H(x,t)}{\partial t} \leq \frac{\partial H(x,t)}{\partial x}R(x,t)\frac{\partial H(x,t)}{\partial x}^{\top}. \quad (6)$$

The condition (6) corresponds to that in Lemma 14 in (Fujimoto and Sugie [2001]) for the time-varying deterministic port-Hamiltonian system. Moreover, suppose that the system is time-invariant. Then the condition (4) always holds. It implies the result in (van der Schaft [1996]) that any time-invariant deterministic port-Hamiltonian system with a positive definite Hamiltonian is passive. This shows that Lemma 4 includes existing results for the deterministic port-Hamiltonian system as a special case.

3. STOCHASTIC GENERALIZED CANONICAL TRANSFORMATIONS AND STABILITY THEOREM

We investigate a special class of feedback transformations of the stochastic port-Hamiltonian system, since the closed-loop system is not necessarily of the form (1). Here we propose stochastic generalized canonical transformations, which are pairs of coordinate and feedback transformations preserving the stochastic Hamiltonian structure. Those are natural extension of the generalized canonical transformations proposed in (Fujimoto and Sugie [2001]) for the deterministic port-Hamiltonian systems.

Definition 2. A set of transformations

$$\bar{x} = \Phi(x,t)\ , \quad \bar{t} = t \tag{7}$$

$$\bar{H} = H(x,t)+U(x,t)\ , \quad \bar{y} = y+\alpha(x,t)\ , \quad \bar{u} = u+\beta(x,t)$$

that changes x to $\bar{x}$, H to $\bar{H}$, y to $\bar{y}$ and u to $\bar{u}$ is said to be a stochastic generalized canonical transformation for the system (1) if it transforms the system into another one which is also described by (1) with another Hamiltonian $\bar{H}$. Here $\Phi(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $U : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $\alpha : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ and $\beta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ are appropriate functions, respectively.

We show the conditions for these transformations.

Theorem 2. Consider the system of the form (1). A set of transformations, functions $\Phi(x,t)$, $U(x,t)$, $\alpha(x,t)$ and $\beta(x,t)$, yields a stochastic generalized canonical transformation defined by (7) if and only if there exists a skew-symmetric matrix $P(x,t)$, a symmetric matrix $Q(x,t)$ such that $R(x,t) + Q(x,t)$ is positive semi-definite, and functions $\Phi(x,t)$, $U(x,t)$ and $\beta(x,t)$ satisfy

$$\frac{\partial \Phi^i}{\partial t} + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\top} hh^{\top}\right\} = \frac{\partial \Phi^i}{\partial x}\left[(J-R)\frac{\partial U}{\partial x}^{\top} + g\beta + (P-Q)\frac{\partial (H+U)}{\partial x}^{\top}\right], \quad (i = 1, 2, \cdots, n). \tag{8}$$

Further a function $\alpha(x,t)$ is given by

$$\alpha(x,t) = g(x,t)^{\top}\frac{\partial U(x,t)}{\partial x}^{\top}. \tag{9}$$

Proof Firstly, the necessity of the theorem is shown. The dynamics of the transformed system in the new coordinate is calculated by utilizing the Itô formula (Itô [1951]) as

$$\begin{aligned}
\mathrm{d}\bar{x}^i &= \frac{\partial \Phi^i}{\partial t}\,\mathrm{d}t + \frac{\partial \Phi^i}{\partial x}\,\mathrm{d}x + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\top} hh^{\top}\right\}\mathrm{d}t \\
&= \left[\frac{\partial \Phi^i}{\partial t} + \frac{\partial \Phi^i}{\partial x}(J-R)\frac{\partial H}{\partial x}^{\top} + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\top} hh^{\top}\right\}\right]\mathrm{d}t \\
&\quad + \frac{\partial \Phi^i}{\partial x} gu\,\mathrm{d}t + \frac{\partial \Phi^i}{\partial x} h\,\mathrm{d}w.
\end{aligned} \tag{10}$$

Suppose that the time-varying stochastic port-Hamiltonian system (1) is transformed into another one using a stochastic generalized canonical transformation with Φ, U and β. Then, the following equation holds for all u and w

$$\begin{aligned}
&\text{R.H.S. of Eq. (10)} \\
&\equiv \left[(\bar{J}-\bar{R})\frac{\partial \bar{H}(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})}{\partial \bar{x}}^{\top}\right]^i \mathrm{d}\bar{t} + \left[\bar{g}\bar{u}\right]^i \mathrm{d}\bar{t} + \left[\bar{h}\,\mathrm{d}w\right]^i \\
&= \frac{\partial \Phi^i}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}(\bar{J}-\bar{R})\left[\frac{\partial \Phi}{\partial x}\right]^{-\top}\frac{\partial (H(x,t)+U(x,t))}{\partial x}^{\top}\mathrm{d}t \\
&\quad + \left[\bar{g}(u+\beta)\right]^i \mathrm{d}t + \left[\bar{h}\,\mathrm{d}w\right]^i.
\end{aligned} \tag{11}$$

Equation (11) and that $\bar{t}$ is identical to t imply that

$$\frac{\partial \Phi}{\partial x} g \equiv \bar{g}, \qquad \frac{\partial \Phi}{\partial x} h \equiv \bar{h}. \tag{12}$$

Using Eqs. (10), (11) and (12), we have

$$\frac{\partial \Phi^i}{\partial t} + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\top} hh^{\top}\right\} = \frac{\partial \Phi^i}{\partial x} \times \left[\left[\frac{\partial \Phi}{\partial x}\right]^{-1}(\bar{J}-\bar{R})\left[\frac{\partial \Phi}{\partial x}\right]^{-\top}\frac{\partial (H+U)}{\partial x}^{\top} - (J-R)\frac{\partial H}{\partial x}^{\top} + g\beta\right]. \tag{13}$$

Here we define the matrices $P(x,t)$ and $Q(x,t)$ as

$$\begin{aligned}
P(x,t) &:= \left[\frac{\partial \Phi}{\partial x}\right]^{-1}\bar{J}(\Phi(x,t),t)\left[\frac{\partial \Phi}{\partial x}\right]^{-\top} - J(x,t), \\
Q(x,t) &:= \left[\frac{\partial \Phi}{\partial x}\right]^{-1}\bar{R}(\Phi(x,t),t)\left[\frac{\partial \Phi}{\partial x}\right]^{-\top} - R(x,t).
\end{aligned} \tag{14}$$

$P(x,t)$ is skew-symmetric, since $J(x,t)$ and $\bar{J}(\Phi(x,t),t)$ are so and, for $R(x,t)$ is symmetric and $\bar{R}(\Phi(x,t),t)$ is symmetric positive semi-definite, $Q(x,t)$ is symmetric and $R(x,t) + Q(x,t)$ is symmetric positive semi-definite. By substituting Eq. (14) for Eq. (13), Eq. (8) is obtained.

The change of the output $\alpha(x,t)$ which yields a stochastic generalized canonical transformation (7) is given by

$$\begin{aligned}
\alpha &= \bar{y} - y = \bar{g}^{\top}\frac{\partial \bar{H}(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})}{\partial \bar{x}}^{\top} - g^{\top}\frac{\partial H(x,t)}{\partial x}^{\top} \\
&= g^{\top}\frac{\partial \Phi(x,t)}{\partial x}^{\top}\left[\frac{\partial \Phi(x,t)}{\partial x}\right]^{-\top}\frac{\partial (H(x,t)+U(x,t))}{\partial x}^{\top} \\
&\quad - g^{\top}\frac{\partial H(x,t)}{\partial x}^{\top} = g^{\top}\frac{\partial U(x,t)}{\partial x}^{\top}.
\end{aligned}$$

This proves the necessity of the theorem.

Secondly, the sufficiency of the theorem is shown. Now suppose the assumption of the theorem holds. Then, by substituting Eq. (8) for (10), the dynamics of the system can be calculated in the new coordinate as

$$\begin{aligned}
\mathrm{d}\bar{x}^i &= \left[\frac{\partial \Phi^i}{\partial x}(J-R)\frac{\partial H}{\partial x}^{\top} + \frac{\partial \Phi^i}{\partial x}\left[(J-R)\frac{\partial U}{\partial x}^{\top} + g\beta\right.\right. \\
&\qquad \left.\left. + (P-Q)\frac{\partial (H+U)}{\partial x}^{\top}\right]\right]\mathrm{d}t + \frac{\partial \Phi^i}{\partial x} gu\,\mathrm{d}t + \frac{\partial \Phi^i}{\partial x} h\,\mathrm{d}w \\
&= \left[\frac{\partial \Phi}{\partial x}(J+P-R+Q)\frac{\partial \Phi}{\partial x}^{\top}\frac{\partial (H+U)}{\partial \bar{x}}^{\top}\Bigg|_{\substack{x = \Phi^{-1}(\bar{x},\bar{t}) \\ t = \bar{t}}}\right]^i \mathrm{d}\bar{t} \\
&\quad + \frac{\partial \Phi^i}{\partial x} g(u+\beta)\,\mathrm{d}\bar{t} + \frac{\partial \Phi^i}{\partial x} h\,\mathrm{d}w.
\end{aligned} \tag{15}$$

$\bar{J}, \bar{R}, \bar{g}$ and $\bar{h}$ are given by

$$\bar{J}(\bar{x},\bar{t}) = \frac{\partial \Phi(x,t)}{\partial x}(J(x,t)+P(x,t))\frac{\partial \Phi(x,t)}{\partial x}^{\top}\Bigg|_{\substack{x = \Phi^{-1}(\bar{x},\bar{t}) \\ t = \bar{t}}}$$

$$\bar{R}(\bar{x},\bar{t}) = \frac{\partial \Phi(x,t)}{\partial x}(R(x,t)+Q(x,t))\frac{\partial \Phi(x,t)}{\partial x}^{\top}\Bigg|_{\substack{x = \Phi^{-1}(\bar{x},\bar{t}) \\ t = \bar{t}}}$$

$$\bar{g}(\bar{x},\bar{t}) = \frac{\partial\Phi(x,t)}{\partial x} g(x,t)\Big|_{\substack{x=\Phi^{-1}(\bar{x},\bar{t})\\ t=\bar{t}}}$$

$$\bar{h}(\bar{x},\bar{t}) = \frac{\partial\Phi(x,t)}{\partial x} h(x,t)\Big|_{\substack{x=\Phi^{-1}(\bar{x},\bar{t})\\ t=\bar{t}}} . \qquad (16)$$

Then, $\bar{J}(\bar{x},\bar{t})$ is skew-symmetric since $J(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})$ and $P(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})$ are so, and $\bar{R}(\bar{x},\bar{t})$ is symmetric positive semi-definite because of the assumption that $R(\Phi^{-1}(\bar{x},\bar{t}),\bar{t}) + Q(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})$ is so. Consequently, the dynamics of the system in the new coordinate is given by

$$\mathrm{d}\bar{x}^i = \left[\left(\bar{J}(\bar{x},\bar{t}) - \bar{R}(\bar{x},\bar{t})\right) \frac{\partial \bar{H}(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})}{\partial \bar{x}}^{\top} \right]^i \mathrm{d}\bar{t} + \left[\bar{g}(u+\beta)\right]^i \mathrm{d}\bar{t} + \left[\bar{h}\,\mathrm{d}w\right]^i. \qquad (17)$$

From Eq. (9), the output in the new coordinate is

$$\bar{y} = g^{\top}\frac{\partial H}{\partial x}^{\top} + g^{\top}\frac{\partial U}{\partial x}^{\top} = g^{\top}\frac{\partial \Phi}{\partial x}^{\top}\left[\frac{\partial\Phi}{\partial x}\right]^{-\top}\frac{\partial (H+U)}{\partial x}^{\top} = \bar{g}^{\top}\frac{\partial \bar{H}(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})}{\partial \bar{x}}^{\top} . \qquad (18)$$

Equations (17) and (18) imply the sufficiency.

Finally, the following theorem states a condition where a transformed stochastic port-Hamiltonian system by a stochastic generalized canonical transformation becomes stochastic passive and, furthermore, the output convergence based on stochastic passivity.

Theorem 3. Consider the system (1) and transform it by an appropriate stochastic generalized canonical transformation such that $\bar{H}(\bar{x},\bar{t}) := H(\Phi^{-1}(\bar{x},\bar{t}),\bar{t}) + U(\Phi^{-1}(\bar{x},\bar{t}),\bar{t}) \geq \bar{H}(0,\bar{t}) = 0$. Then, the transformed system becomes stochastic passive with new Hamiltonian $\bar{H}(\bar{x},\bar{t})$ as a storage function if and only if the following inequality holds:

$$-\frac{\partial(H+U)}{\partial x}\left[\frac{\partial\Phi}{\partial x}\right]^{-1}\frac{\partial\Phi}{\partial t} + \frac{\partial(H+U)}{\partial t} + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial(H+U)}{\partial x}\left[\frac{\partial\Phi}{\partial x}\right]^{-1}\right)^{\top} h(x)h(x)^{\top}\frac{\partial\Phi}{\partial x}^{\top}\right\} \leq \frac{\partial(H+U)}{\partial x}(R+Q)\frac{\partial(H+U)}{\partial x}^{\top} . \qquad (19)$$

Furthermore, suppose that $\lim_{|\bar{x}|\to\infty}\inf_{0\leq\bar{t}<\infty}\bar{H}(\bar{x},\bar{t}) = \infty$ and if one of the following conditions holds:

(i) for each initial state $\bar{x}_0$ there is a $d > 2$ such that
$$\sup_{0\leq\bar{t}<\infty} E[|\bar{x}(\bar{t})|^d] < \infty. \qquad (20)$$
(ii) $\bar{h}(\bar{x},\bar{t})$ is bounded.
(iii) Almost every sample path of $\int_0^{\bar{t}}\bar{h}(\bar{x}(\tau),\tau)\,\mathrm{d}w(\tau)$ is uniformly continuous on $\bar{t}\geq 0$.

Then under the unity feedback $\bar{u} = -\bar{y}$, for every initial state $\bar{x}_0 \in \mathbb{R}^n$, $\lim_{\bar{t}\to\infty}\bar{H}(\bar{x},\bar{t})$ exists and is finite almost surely and, moreover, $\lim_{\bar{t}\to\infty}\bar{y}(\bar{t}) = 0$ holds almost surely.

Proof Firstly, the former part of the theorem is shown. Due to Lemma 1, the necessary and sufficient condition is that the following inequality holds in the new coordinate

$$\frac{\partial\bar{H}(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})}{\partial\bar{t}} + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{H}(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})}{\partial\bar{x}}\right)^{\top}\bar{h}\bar{h}^{\top}\right\} \leq \frac{\partial\bar{H}}{\partial\bar{x}}\bar{R}\frac{\partial\bar{H}}{\partial\bar{x}}^{\top} . \qquad (21)$$

The first term in the left hand side of (21) is calculated as

$$\frac{\partial\bar{H}(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})}{\partial\bar{t}} = \frac{\partial(H+U)}{\partial x}\frac{\partial\Phi^{-1}(\bar{x},\bar{t})}{\partial\bar{t}} + \frac{\partial(H+U)}{\partial t}. \qquad (22)$$

Since the Jacobian of the pair of the coordinate transformations $\bar{x} = \Phi(x)$ and $\bar{t} = t$, denoted by $\mathcal{J}$, is given by

$$\mathcal{J} = \begin{pmatrix} \frac{\partial\Phi(x,t)}{\partial x} & \frac{\partial\Phi(x,t)}{\partial t} \\ O_{1n} & 1 \end{pmatrix},$$

the Jacobian of the inverse coordinate transformation which coincides with $\mathcal{J}^{-1}$ is obtained as

$$\mathcal{J}^{-1} = \begin{pmatrix} \left[\frac{\partial\Phi}{\partial x}\right]^{-1} & -\left[\frac{\partial\Phi}{\partial x}\right]^{-1}\frac{\partial\Phi}{\partial t} \\ O_{1n} & 1 \end{pmatrix} \equiv \begin{pmatrix} \frac{\partial\Phi^{-1}}{\partial\bar{x}} & \frac{\partial\Phi^{-1}}{\partial\bar{t}} \\ O_{1n} & 1 \end{pmatrix}, \qquad (23)$$

where O_{ij} denotes $i \times j$ zero matrix. It follows from Eq. (23) that

$$\frac{\partial\Phi^{-1}(\bar{x},\bar{t})}{\partial\bar{t}} = -\left[\frac{\partial\Phi(x,t)}{\partial x}\right]^{-1}\frac{\partial\Phi(x,t)}{\partial t}. \qquad (24)$$

Here let us note that the following equation holds

$$\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{H}}{\partial\bar{x}}\right)^{\top} = \frac{\partial}{\partial x}\left(\frac{\partial\bar{H}}{\partial x}\frac{\partial\Phi^{-1}}{\partial\bar{x}}\right)^{\top}\frac{\partial\Phi^{-1}}{\partial\bar{x}} = \frac{\partial}{\partial x}\left(\frac{\partial\bar{H}}{\partial x}\left[\frac{\partial\Phi}{\partial x}\right]^{-1}\right)^{\top}\left[\frac{\partial\Phi}{\partial x}\right]^{-1} . \qquad (25)$$

Substituting Eq. (24) for Eq. (22), equations (16),(21),(22) and (25) imply that Eq. (19) holds immediately.

Secondly, the latter part of the theorem is shown. Since the transformed system is stochastic passive, the following inequality holds for the closed loop system with the unity feedback $\bar{u} = -\bar{y}$: $\mathcal{L}\bar{H}(\bar{x},\bar{t}) \leq -\|\bar{y}\|^2 \leq 0$. Then the rest of the theorem is shown by directly applying Theorem 2.1 and Theorem 2.5 in (Mao [1999]).

4. NUMERICAL EXAMPLE

In this section, we consider stabilization of a rolling coin on a horizontal plane (van der Schaft [1996], Fujimoto and Sugie [1999], Bloch [2003]) depicted in Fig. 1 in the presence of noise. Let X-Y denote the orthogonal coordinates of the point of contact of the coin. Let q^1 denote the heading angle of the coin, and (q^2, q^3) the position of the coin in X-Y plane. Furthermore let p^1 be

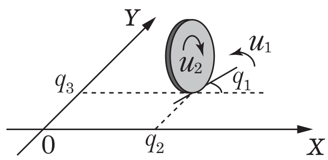

Fig. 1. A rolling coin

the angular velocity with respect to the heading angle, p^2 be the rolling angular velocity of the coin, u^1 and u^2 be the accelerations with respect to p^1 and p^2, respectively.

Finally, let all the parameters unity for simplicity. Then this system is described by a stochastic port-Hamiltonian system of the form (1) with $q = (q^1, q^2, q^3)^\top$, $p = (p^1, p^2)^\top$, $x = (q^\top, p^\top)^\top$, $H(x) = (1/2)p^\top p$ and

$$J(x,t) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cos q^1 \\ 0 & 0 & 0 & 0 & \sin q^1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -\cos q^1 & -\sin q^1 & 0 & 0 \end{pmatrix},$$

$$R(x,t) = O_{55} \quad , \quad g(x,t) = \begin{pmatrix} O_{32} \\ I_2 \end{pmatrix},$$

$$h(x,t) = \begin{pmatrix} O_{32} \\ \mathrm{diag}\{h^1(x,t), h^2(x,t)\} \end{pmatrix} \quad , \quad y = p, \qquad (26)$$

where I_i denotes $i \times i$ identity matrix, and $h^1(x,t)$ and $h^2(x,t)$ represent appropriate functions for the noise port.

The literature (Fujimoto and Sugie [2001]) has proposed time-varying asymptotically stabilizing controllers for deterministic nonholonomic Hamiltonian systems. In this method, a special class of time-varying generalized canonical transformations is introduced in which a time-varying potential function $U(x,t)$ is parameterized by an arbitrary periodic function $\alpha(q,t)$ which is the parameter of the change of the output. The purpose of this section is to design a time-varying feedback controller which renders the origin asymptotically stable in probability based on the framework in (Fujimoto and Sugie [2001]).

Firstly, we consider the following form of the new Hamiltonian $\bar{H} = \frac{1}{2}(p+\alpha)^\top(p+\alpha) + V(\bar{q})$, where $\alpha(q,t)$ is any periodic odd function and an appropriate function $V(\bar{q})$ should be chosen so that $\bar{H}$ is non-negative in the new coordinate. Here we utilize the following functions which are the same as those in (Fujimoto and Sugie [2001])

$$\alpha(q,t) = \begin{pmatrix} q^3 \sin t \\ 0 \end{pmatrix} \quad , \quad V(\bar{q}) = \frac{1}{2}\bar{q}^\top K \bar{q}, \qquad (27)$$

where K is defined as $K := \mathrm{diag}\{k^1, k^2, k^3\}$ with appropriate positive numbers k^1, k^2 and k^3. Then let us construct the time-varying stochastic generalized canonical transformation with the following coordinate transformation utilized in (Fujimoto and Sugie [2001])

$$\bar{q} = \begin{pmatrix} q^1 - q^3 \cos t \\ q^2 \\ q^3 \end{pmatrix} \quad , \quad \bar{p} = \begin{pmatrix} p^1 + q^3 \sin t \\ p^2 \end{pmatrix}. \qquad (28)$$

In order to obtain a time-varying stochastic generalized canonical transformation, let us decide the rest design parameters $\beta(x,t) = (\beta^1, \beta^2)^\top, P(x,t)$ and $Q(x,t)$ by utilizing Theorem2. The following choice satisfies Eq. (8)

$$P(x,t) = O_{55} \ , \ Q(x,t) = \left(\begin{array}{c|cc} O_{33} & \multicolumn{2}{c}{O_{32}} \\ \hline O_{23} & Q_{44}(x,t) & 0 \\ & 0 & Q_{55}(x,t) \end{array}\right) \qquad (29)$$

$$\begin{aligned} \beta^1(x,t) &= q^3 \cos t + k^1(q^1 - q^3 \cos t) + p^2 \sin t \sin q^1 \\ &\quad + (p^1 + q^3 \sin t) Q_{44}(x,t) \\ \beta^2(x,t) &= -k^1(q^1 - q^3 \cos t) \sin q^1 \cos t + k^2 q^2 \cos q^1 \\ &\quad + k^3 q^3 \sin q^1 + p^2 Q_{55}(x,t), \end{aligned} \qquad (30)$$

where free parameters $Q_{44}(x,t)$ and $Q_{55}(x,t)$ should be chosen so that $Q(x,t)$ becomes symmetric positive semi-definite. Furthermore in order to obtain stochastic passivity, let us derive another condition from Theorem 3. It follows from the inequality (19) that

$$(p^1 + q^3 \sin t)^2 Q_{44} + (p^2)^2 Q_{55} \geq \frac{h^1(x,t)^2 + h^2(x,t)^2}{2}. \qquad (31)$$

In what follows, we suppose that there exist functions $Q_{44}(x,t)$ and $Q_{55}(x,t)$ such that the equality in the condition (31) holds and $Q(x,t)$ in Eq. (29) becomes symmetric positive semi-definite.

The transformed system is given by

$$\bar{J}(\bar{x},\bar{t}) = \begin{pmatrix} 0 & 0 & 0 & 1 & -\sin q^1 \cos t \\ 0 & 0 & 0 & 0 & \cos q^1 \\ 0 & 0 & 0 & 0 & \sin q^1 \\ -1 & 0 & 0 & 0 & \sin q^1 \sin t \\ \sin q^1 \cos t & -\cos q^1 & -\sin q^1 & -\sin q^1 \sin t & 0 \end{pmatrix},$$

$$\bar{R}(\bar{x},\bar{t}) = Q(x,t) \quad , \quad \bar{g}(\bar{x}) = \begin{pmatrix} O_{32} \\ I_2 \end{pmatrix}, \qquad (32)$$

$$\bar{h}(\bar{x}) = \left.\begin{pmatrix} O_{32} \\ \mathrm{diag}\{h^1(x,t), h^2(x,t)\} \end{pmatrix}\right|_{\substack{x = \Phi^{-1}(\bar{x},\bar{t}) \\ t = \bar{t}}} \quad , \quad \bar{y} = \bar{p}.$$

Equations (32) implies that the transformed system has the form of (1). Since this system obtains stochastic passivity, it can be easily proven by Theorem 3 that $\lim_{\bar{t}\to\infty} \bar{y} = 0$ almost surely with the unity feedback $\bar{u} = -\bar{y}$. Generally Theorem 3 only guarantees that the convergence of the output. However, in this case, we can show that the unity feedback also renders the origin of the system (26) asymptotically stable almost surely. Let $\bar{u} = \bar{y} = \bar{p} \equiv 0$ of the system (32). Then it follows from Eq. (28) and the condition (31) that $h(x,t) \equiv 0$. The literature (Fujimoto and Sugie [2001]) has proven that the transposed system (32) has the zero-state observability with respect to x without noise. These two facts prove the claim. Eventually, we obtain the following time-varying feedback controller which renders the origin asymptotically stable in probability as

$$u = -\beta(x,t) - (y + \alpha(x,t)) \qquad (33)$$

(see, Eqs.(27) and (30) for $\alpha(x,t)$ and $\beta(x,t)$).

Finally, let us show some simulation results. Here we consider the noise port as $h^1(x,t) = 0$ and $h^2(x,t) = k^2 p^2$ and we set a matrix K in (27) as $K = \mathrm{diag}\{1,1,1\}$, design parameters as $Q_{44} = 0$ and $Q_{55} = (k^2)^2/2$ with $k^2 = 8$ and the initial condition as $(q^1, q^2, q^3, p^1, p^2) = (0, 0, 1.0, 0, 0)$. We simulate a standard Wiener process in the same manner as in (Ohsumi [2002]).

Firstly, we consider a scenario where there exists noise and a feedback controller designed by a deterministic method in (Fujimoto and Sugie [2001]), which corresponds to the case where $Q(x,t) \equiv 0$, is applied. Figure 2 shows the motion of the coin in X-Y plane and Fig. 3 shows time responses of q and p. They imply that the behavior of the system seems unstable with the controller for the deterministic system in the presence of noise. Secondly, we consider a scenario where there exists the same noise as in the previous scenario and the feedback controller (33) designed by the proposed method is applied. Figures 4 and 5 imply that the proposed controller works well even in the presence of noise. Because of equipping time-varying feedback controllers, the convergence is slow and oscillatory (Pomet [1992], Fujimoto and Sugie [2001]).

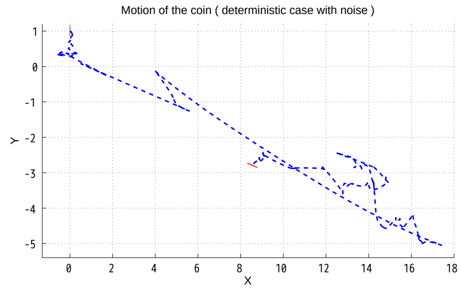

Fig. 2. Motion of the coin in deterministic case with noise

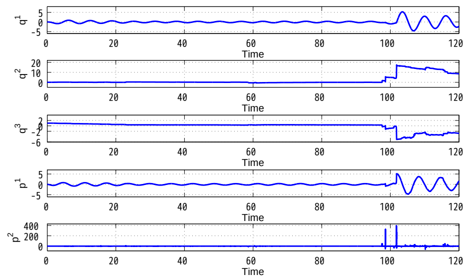

Fig. 3. q and p in deterministic case with noise

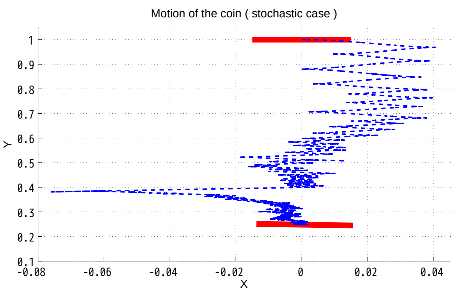

Fig. 4. Motion of the coin in stochastic case

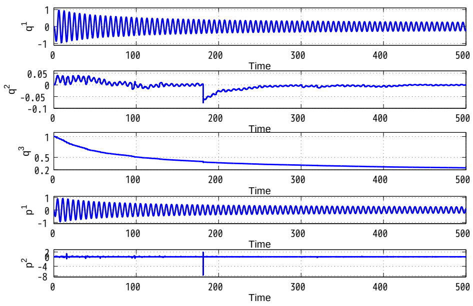

Fig. 5. q and p in stochastic case

These simulation results demonstrate the effectiveness of the proposed framework.

5. CONCLUSION

This paper has introduced time-varying stochastic port-Hamiltonian systems and has clarified some of their properties. We have extended the authors' previous results for the time-invariant stochastic port-Hamiltonian systems in (Satoh and Fujimoto [2008a]) to the time-varying case. Now we tackle stochastic trajectory tracking control problem by applying the proposed method here in (Satoh and Fujimoto [2008b]).

REFERENCES

Bloch, A.M. (2003). *Nonholonomic Mechanics and Control*. Springer-Verlag, New York.

Brockett, R.B. (1983). Asymptotic stability and feedback stabilization. In R.W. Brockett, R.S. Millmann, and H.J. Sussmann (eds.), *Differential Geometric Control Theory*, 181–191. Birkhäuser, Basel-Boston.

Bucy, R.S. (1965). Stability and positive supermartingales. *J. Differential Equations*, 1(2), 151–155.

Byrnes, C.I., Isidori, A., and Willems, J.C. (1991). Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems. *IEEE Trans. Autom. Contr.*, 36(11), 1228–1240.

Florchinger, P. (1999). A passive system approach to feedback stabilization of nonlinear control stochastic systems. *SIAM J. Control Optim.*, 37(6), 1848–1864.

Fujimoto, K. and Sugie, T. (1999). Stabilization of a class of Hamiltonian systems with nonholonomic constraints via canonical transformations. In *Proc. European Control Conference*.

Fujimoto, K. and Sugie, T. (2001). Canonical transformation and stabilization of generalized Hamiltonian systems. *Systems & Control Letters*, 42(3), 217–227.

Has'minskii, R.Z. (1980). *Stochastic Stability of Differential Equations*. Sijthoff and Noordhoff, Maryland, USA.

Ikeda, N. and Watanabe, S. (1989). *Stochastic differential equations and diffusion processes*. North-Holland/Kodansha, Amsterdam/Tokyo, second edition.

Itô, K. (1951). On a formula concerning stochastic differentials. *Nagoya Math. J.*, 3, 55–65.

Kushner, H.J. (1967). *Stochastic Stability and Control*. Academic Press.

Mao, X. (1991). *Stability of Stochastic Differential Equations with Respect to Semimartingales*. Pitman Res. Notes Math. Ser. 251. Longman Scientific and Technical.

Mao, X. (1999). Stochastic versions of the LaSalle theorem. *J. Differential Equations*, 153, 175–195.

Maschke, B. and van der Schaft, A.J. (1992). Port-controlled Hamiltonian systems: modelling origins and system theoretic properties. In *Proc. 2nd IFAC Symp. Nonlinear Control Systems*, 282–288.

Ohsumi, A. (2002). *Introduction to Stochastic Systems*. Asakura Publishing, Tokyo. (in Japanese).

Øksendal, B. (1998). *Stochastic differential equations, An introduction with applications*. Springer-Verlag, Berlin Heidelberg New York, fifth edition.

Pomet, J.B. (1992). Explicit design of time-varying stabilizing control laws for a class of controllable systems without drift. *Systems & Control Letters*, 18, 147–158.

Sastry, S. (1999). *Nonlinear Systems: Anaysis, Stability and Control*, volume 10 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York.

Satoh, S. and Fujimoto, K. (2008a). On passivity based control of stochastic port-hamiltonian systems. In *Proc. 47th IEEE Conf. on Decision and Control*, 4951–4956.

Satoh, S. and Fujimoto, K. (2008b). Stabilization of time-varying stochastic port-Hamiltonian systems and its application to stochastic trajectory tracking control. In *Proc. 37th SICE Symposium on Control Theory*, 393–398. (in Japanese).

van der Schaft, A.J. (1996). *L_2-gain and Passivity Techniques in Nonlinear Control*, volume 218. Lecture Notes on Control and Information Science, Berlin.