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Stochastic stabilization of rigid body motion of a spacecraft on SE(3)

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ABSTRACT
In this paper, we consider a stochastic systems modeling and stabilization of the rigid body motion of a spacecraft with stochastic disturbances. To rigorously deal with a continuous time revolution disturbed by a stochastic process, in this paper, the rigid body motion of the spacecraft is described by a stochastic differential equation (SDE) on $SE(3)$. Stochastic systems modeling with SDE enables us to execute the stochastic analysis, and to quantitatively evaluate the uncertainty. We present a stochastic stabilizing controller and a stability theorem, which claim that the error of the rigid body motion with respect to a given desired motion is exponentially ultimately bounded in the mean square sense. Note that the resultant stochastic rigid body motion model on $SE(3)$ has no equilibrium point due to persistent noise effect. This makes stability analysis more difficult than the conventional stochastic stability concepts. However, the present stability theorem guarantees that the error exponentially converges to the vicinity of the target state and then remains bounded even under persistent noise. Finally, numerical simulations demonstrate the effectiveness of the proposed method.

KEYWORDS
Nonlinear stochastic systems; Stochastic stability; Special Euclidean group $SE(3)$; Exponentially ultimately boundedness

1. Introduction

Stochastic stability analysis and stabilization (Has'minskii, 1980; Kushner, 1967; Mao, 1990) are utilized for rigorously ensuring stability of nonlinear systems with stochastic uncertainty. Particularly, stabilization of the attitude and position of a spacecraft is necessary for many practical missions such as high resolution observation, highly efficient communication, high precision rendezvous docking, and so on. The major obstacles of stabilization are internal and external uncertainties possibly caused by measurement and communication noises, environmental disturbances including solar radiation pressure, gravity gradient, magnetic torques and atmospheric drag, and modeling errors due to misalignments and secular changes of the spacecraft. We cannot estimate these uncertainties accurately in advance, since they include more or less stochastic elements.

Although there have been proposed many stabilization methods, most of them consider a spacecraft under external disturbances as a deterministic control system, and they suppose that external disturbances are partially known, unknown but have upper

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bounds, or vanish at an equilibrium point. The literature (Kristiansen, Nicklasson, & Gravdahl, 2009) presents an attitude tracking controller using quaternion feedback and integral backstepping. Although the control method is adapted to the European Student Earth Orbiter in the presence of disturbances, it deals with known disturbances. Also, the attitude tracking and disturbance rejection problem is considered in (Chen & Huang, 2009), where the external disturbance is supposed to be sinusoidal disturbance with unknown amplitudes and phase angles, but known frequencies. Besides, in (Ding & Li, 2009), a discontinuous finite-time control law is proposed based on terminal sliding mode control, which drives the state of a rigid spacecraft to a neighborhood of the desired state. A backstepping based adaptive sliding mode control is employed in (Cong, Liu, & Chen, 2013) to compensate for inertia matrix uncertainty and external disturbances. However, both references (Cong et al., 2013; Ding & Li, 2009) assume the upper bound of the external disturbances. The literature (Bai, Biggs, Wang, & Cui, 2017) presents an attitude tracking control with adaptive parameter updates to compensate for disturbances with known bound. Although it also proposes an alternative approach for unknown disturbances, disturbance estimation using the extended state observer is instead required. Since those methods deal with deterministic bounded signals as disturbances, they cannot deal with stochastic uncertainties nor unbounded signals such as white noise. Moreover, they only consider disturbances of the dimension of the generalized force, which are directly compensated by the control forces and/or torques. The fluctuations in the kinematics, e.g., noisy attitude information are not taken into account.

Regarding this point, this paper models the rigid body motion of a spacecraft under internal and external uncertainties as a stochastic system, whose kinematics is described by a stochastic differential equation (SDE) (Gihman & Skorohod, 1972; Ikeda & Watanabe, 1992; Øksendal, 1998). Then, we design a stabilizing controller and prove a stochastic stability theorem, where stochastic uncertainties are rigorously taken into account via stochastic calculus. Stability analysis using stochastic Lyapunov functions is a powerful scheme for a nonlinear stochastic system, since the analysis can be executed without using the solution process of the system, namely, a solution to a nonlinear SDE, which is analytically intractable in general. The stochastic controller can tackle a robust stabilization problem under stochastic disturbances driven by white noise. White noise itself contains all frequency components, and can generate other stochastic signals with specific frequency characteristics through appropriate shaping filters. Moreover, white noise is unbounded, and can be arbitrarily large at any finite interval. This makes stability analysis and controller design challenging.

Stochastic stabilization or estimation problems of the attitude and/or position of a spacecraft with stochastic disturbances have been investigated by several researchers so far. Most of preliminary works consider the motion of a spacecraft on the special orthogonal group $SO(3)$ or the special Euclidean group $SE(3)$. In (Samiei, Izadi, Viswanathan, Sanyal, & Butcher, 2015), a stochastic feedback control of the attitude dynamics of a spacecraft is considered. However, a stochastic disturbance is only considered in dynamics not in kinematics. Essential difficulty of stochastic analysis and control appears in kinematics, since the rotational kinematics is described on the manifold $SO(3)$, which is not homeomorphic to the Euclidean space. In (Choukroun, 2009), stochastic modeling of the attitude quaternion kinematics of a spacecraft with state-multiplicative noise and an optimal quaternion filter is discussed. Although stochastic uncertainty on kinematics is considered, the control of the attitude of a spacecraft is not taken into account. Besides, the quaternion representation is free from the singularity problem unlike the Euler angles representation, while it may suffer from the unwinding phenomenon (Bhat & Bernstein, 2000), which is caused by the fact that two antipodes represent the same attitude. Moreover, it cannot deal with both translational and rotational motions, namely, the rigid body motion.

For the rigid body motion stabilization or estimation, the matrix Lie group representations have been utilized in, e.g., (Chaturvedi, Sanyal, & McClamroch, 2011; Cunha, Silvestre, & Hespanha, 2008; Lee, 2015; Mahony, Hamel, & Pflimlin, 2008; Roza & Maggiore, 2012; Solo, 2012; Yamauchi, Nakano, Hatanaka, & Fujita, 2014; Yamauchi, Nakano, Hatanaka, Fujita, & Satoh, 2015). In (Solo, 2012), an attitude estimation problem of a spacecraft on $SO(3)$ is proposed. Here, the attitude estimator is constructed using a stochastic differential equation on $SO(3)$, and stochastic stability of the estimation error is proved based on stochastic Lyapunov analysis. Also, a stochastic optimal motion planning for the attitude kinematics of a rigid body is proposed in (Lee, 2015). The paper solves the Fokker-Planck equation using noncommutative harmonic analysis, and the optimal trajectory of a rigid body while avoiding obstacles is provided by formulating a stochastic optimal control problem. Both references (Lee, 2015; Solo, 2012), however, do not ensure stability of the resultant attitude motion, nor do they deal with the translational motion. On the contrary, in the literature (Chaturvedi et al., 2011; Cunha et al., 2008; Mahony et al., 2008; Roza & Maggiore, 2012; Yamauchi et al., 2014, 2015), the rigid body motion is treated on the special Euclidean group $SE(3)$. In (Cunha et al., 2008), a Lyapunov-based output-feedback controller for the attitude and position control is constructed on $SE(3)$, which guarantees almost global asymptotic stability of the desired equilibrium point. The literature (Roza & Maggiore, 2012) proposes an almost globally asymptotically stabilizing controller for a vehicle to achieve the desired attitude and position based on the classical backstepping method. However, both methods do not consider any uncertainty or disturbance. In contrast, the literature (Yamauchi et al., 2014, 2015) deals with the rigid body motion with stochastic uncertainty. Here, the authors consider the performance analysis of a visual motion observer, where the noncooperative target is supposed to move randomly. Mean-square stability of the estimation error is analyzed based on a stochastic system modeling on $SE(3)$. However, the estimation is only considered, and thus control of the rigid body motion under uncertainty is not considered.

The main contribution of this paper is to present a stochastic stabilizing controller and a stability theorem, which guarantee that the rigid body kinematics of a spacecraft with stochastic disturbances is stabilized to a desired attitude and position. The difficulties in the stability analysis arise from two causes. One is that the state space is described by a matrix Lie group on manifold $SE(3)$, and we model the rigid body kinematics with uncertainties using Itô stochastic differential equation of a matrix form. The other is that the resultant stochastic rigid body kinematics model on $SE(3)$ has no equilibrium point due to persistent noise effect. Thus, the conventional stochastic stability analysis in (Has'minskii, 1980; Kushner, 1967) cannot be directly applied. In order to conduct stability analysis of a stochastic system with no equilibrium point, we employ the stability concept of exponentially ultimately boundedness in mean square sense, and its sufficient condition using stochastic Lyapunov-like function (Miyahara, 1972; Xie & Khargonekar, 2012). Finally, numerical example demonstrates the effectiveness of the proposed method. This paper focuses on the rigid body kinematics, since fundamental difficulties come from the geometry of the stochastic kinematics model. However, the proposed method is readily applied for the rigid body motion dynamics of a spacecraft as well.

2. Preliminaries

2.1. Stabilization of nonlinear stochastic systems

In this section, we introduce continuous-time nonlinear stochastic systems (Gihman & Skorohod, 1972; Ikeda & Watanabe, 1992; Øksendal, 1998) and review some basics on the stochastic stability analysis (Has'minskii, 1980; Kushner, 1967; Mao, 1990). Here, we consider a stochastic system described by Itô SDE as follows:

$$\mathrm{d}x = f(x)\mathrm{d}t + h(x)\mathrm{d}w, \quad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^r$ is a Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ satisfying $\mathbb{E}[\mathrm{d}w(t)\mathrm{d}w(t)^{\mathrm{T}}] = \Sigma\Sigma^{\mathrm{T}}\mathrm{d}t, \forall t \in \mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0} := [0, \infty)$. Here, $\mathbb{E}[\cdot]$ denotes the expectation with respect to the probability measure $\mathcal{P}$, and the matrix $\Sigma\Sigma^{\mathrm{T}} \in \mathbb{R}^{r\times r}$ represents the covariance matrix. Also, Ω is a sample space and $\mathcal{F}$ is a sigma algebra of the observable random events and $\mathcal{P}$ is a probability measure on Ω. We suppose that the function $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^{n\times r}$ are sufficiently differentiable functions, and the existence and uniqueness of a solution to the system (1) is assumed.

Then, we proceed to the stochastic stability analysis. First, we define the following infinitesimal operator $\mathcal{L}(\cdot)$ in order to describe the time evolution of a scalar function along with the system (1):

$$\mathcal{L}(\cdot) := \frac{\partial(\cdot)}{\partial x} f + \frac{1}{2}\mathrm{tr}\left\{ \frac{\partial}{\partial x}\left(\frac{\partial(\cdot)}{\partial x}\right)^{\mathrm{T}} h\Sigma\Sigma^{\mathrm{T}}h^{\mathrm{T}} \right\}, \tag{2}$$

where $\mathrm{tr}\{\cdot\}$ represents the trace of the argument. While the first term of right hand side of Eq. (2) denotes the Lie derivative along the vector field f, the second term represents the influence of probabilistic uncertainties, which is a specific term in stochastic calculus. Therefore, stochastic calculus enables one to quantitatively take uncertainties into account in stability analysis and controller design. Let $\Phi(x) : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function. Then, Itô formula (Itô, 1951) provides its expected infinitesimal variation as

$$\mathbb{E}[\mathrm{d}\Phi(x)] = \mathbb{E}[\mathcal{L}(V(x))]\mathrm{d}t + o(\mathrm{d}t), \tag{3}$$

where $o(\cdot)$ denotes a term satisfying $\lim_{\epsilon\to 0} o(\epsilon)/\epsilon = 0$. The conventional stochastic stability analysis, e.g., (Has'minskii, 1980; Kushner, 1967; Mao, 1990) assumes that the system has an equilibrium point, where the noise vanishes. However, the case that the equilibrium point does not exist is theoretically and practically important, where the state does not converge to any target point, and it fluctuates around the point. Such a case arises, for example, due to persistent noise or geometric structure of the state space. The latter case will be encountered in this paper. In order to deal with the case, this paper employs the following stability concept:

Definition 2.1. (Miyahara, 1972; Xie & Khargonekar, 2012) Consider the stochastic system (1). The system is said to be exponentially ultimately bounded in mean square sense if there exist positive constants c_1, c_2 and c_3 such that the following condition

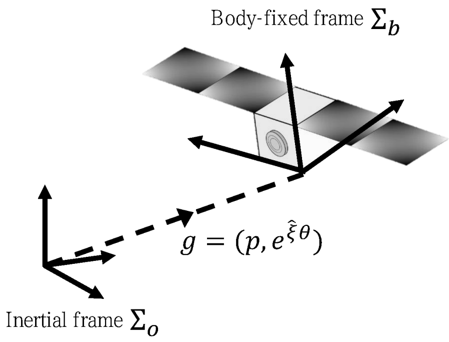

Figure 1. Coordinate frame

holds:

$$\mathbb{E}[\|x(t)\|^2] \leq c_1 e^{-c_2 t} + c_3 \qquad \forall t \geq 0.$$

This means that the expectation of the squared norm of the state $x(t)$, namely, $\mathbb{E}[\|x(t)\|^2]$ decreases exponentially until a certain time, and then remains under the upper bound c_3. The next theorem provides sufficient conditions for the exponential boundedness in mean square sense.

Theorem 2.2. *(Miyahara, 1972; Xie & Khargonekar, 2012) Consider the stochastic system (1). Assume that there exist a twice differentiable function $V(x)$ and constants $c_1, k_1 > 0$ and $k_2 \geq 0$ such that*

$$\begin{aligned} &c_1\|x\|^2 \leq V(x), \\ &\mathcal{L}(V(x)) \leq -k_1 V(x) + k_2. \end{aligned}$$

Suppose that $\mathbb{E}[V(x_0)] < \infty$ holds. Then, the system is exponentially ultimately bounded in mean square sense. Moreover, an ultimate upper bound c_3 in Definition 2.1 is given by $c_3 = k_2/(c_1 k_1)$.

2.2. Kinematics of a rigid body spacecraft

In this section, we briefly review the kinematics of a rigid body spacecraft. See the reference (Murray, Li, & Sastry, 1994) for more details. Suppose that the spacecraft moves in $\mathbb{R}^3$. As shown in Figure 1, we denote an inertial frame and a body-fixed frame by Σ_o and Σ_b, respectively. Let $p \in \mathbb{R}^3$ and $e^{\hat{\xi}\theta} \in \mathbb{R}^{3\times 3}$ be the position vector and the rotation matrix of the frame Σ_b relative to the frame Σ_o, where $\xi \in \mathbb{R}^3$ specifies the instantaneous axis of rotation and θ denotes the rotation angle. The operator $\hat{(\cdot)}$ is such that $\hat{a}b = a \times b$ for any vector $a, b \in \mathbb{R}^3$ with the vector cross product $\times$, namely, $\hat{a}$ denotes a skew-symmetric matrix. The rotation matrix $e^{\hat{\xi}\theta}$ is defined on the special orthogonal matrix Lie group:

$$SO(3) = \left\{ e^{\hat{\xi}\theta} \in \mathbb{R}^{3\times 3} : e^{\hat{\xi}\theta} {e^{\hat{\xi}\theta}}^{\mathrm{T}} = I_3, \det\left\{e^{\hat{\xi}\theta}\right\} = +1 \right\},$$

where $I_n \in \mathbb{R}^{n\times n}$ is the $n\times n$ identity matrix, and $\det\{\cdot\}$ stands for the determinant of a square matrix. Then, we consider the translational motion of the spacecraft as well as the rotational motion. The configuration space of the system is the product space of $\mathbb{R}^3$ with $SO(3)$, which is denoted as $SE(3)$:

$$SE(3) = \left\{(p, e^{\widehat{\xi}\theta}) : p \in \mathbb{R}^3, e^{\widehat{\xi}\theta} \in \mathbb{R}^{3\times 3}\right\} = \mathbb{R}^3 \times SO(3).$$

In what follows, we employ the following 4×4 matrix

$$g := \begin{bmatrix} e^{\widehat{\xi}\theta} & p \\ 0 & 1 \end{bmatrix} \tag{4}$$

as the homogeneous representation of $g = (p, e^{\widehat{\xi}\theta}) \in SE(3)$. We next define the homogeneous representation of the body velocity of a rigid motion $g \in SE(3)$ as follows:

$$\widehat{V} := g^{-1}\dot{g} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}, \tag{5}$$

where $\omega \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ denote the instantaneous body angular velocity and the velocity of the origin of Σ_b relative to Σ_o. From Eq. (5), the rigid body motion g obeys the following ordinary differential equation:

$$\dot{g} = g\widehat{V}.$$

3. Stabilization of the rigid body motion of the spacecraft

In this section, we formulate a stochastic model of the kinematics of the spacecraft and analyze stability of the rigid body motion. Here, we assume that the body velocity of the spacecraft in Eq. (5) can be directly controlled and that stochastic uncertainty exists in the same dimension as the body velocity. Let the white Gaussian noise $\gamma = [{\gamma_\omega}^{\mathrm{T}}, {\gamma_v}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^6$ be a set of uncertainties with the covariance matrix $\Sigma\Sigma^{\mathrm{T}}$ given by

$$\Sigma := \begin{bmatrix} \Sigma_\omega \\ \Sigma_v \end{bmatrix}, \quad \Sigma_\omega, \Sigma_v \in \mathbb{R}^{3\times r},$$

where a positive constant r denotes the dimension of a set of independent standard white Gaussian noises. Under the prescribe setting, we begin with the following ordinal differential equation:

$$\dot{e}^{\widehat{\xi}\theta} = e^{\widehat{\xi}\theta}(\widehat{u_R} + \widehat{\gamma_\omega}),$$

which is intuitive but not mathematically rigorous. Here, we write ω as u_R to make it clear that the body angular velocity is supposed to be the control input. There are two representations commonly used to describe the stochastic differential equations, called Stratonovich SDE and Itô SDE (Øksendal, 1998). In order to ensure a rigorous formulation of a differential equation with a white Gaussian noise, we first build a stochastic system using Stratonovich SDE based on the Wong-Zakai approximation theorem (Wong & Zakai, 1965) as

$$\mathrm{d}e^{\widehat{\xi}\theta} = e^{\widehat{\xi}\theta}\widehat{u_R}\mathrm{d}t + e^{\widehat{\xi}\theta} \circ \widehat{\mathrm{d}w_\omega}, \tag{6}$$

where $w_\omega \in \mathbb{R}^3$ denotes a Wiener process satisfying $\mathbb{E}[\mathrm{d}w_\omega \mathrm{d}w_\omega^{\mathrm{T}}] = \Sigma_\omega \Sigma_\omega^{\mathrm{T}} \mathrm{d}t$. In order to investigate stochastic stability based on stochastic calculus, we transform the Stratonovich SDE (6) into the almost sure equivalent Itô SDE. By adding the Wong-Zakai correction term to (6), we obtain the following stochastic system model described by the Itô SDE:

$$\mathrm{d}e^{\widehat{\xi}\theta} = e^{\widehat{\xi}\theta}(-A_1(\Sigma_\omega) + \widehat{u_R})\mathrm{d}t + e^{\widehat{\xi}\theta}\widehat{\mathrm{d}w_\omega}. \tag{7}$$

The Wong-Zakai correction term denoted by $A_1(\Sigma_\omega)$ is given by

$$A_1(\Sigma_\omega) := \frac{1}{2}\mathrm{tr}\left\{\Sigma_\omega {\Sigma_\omega}^{\mathrm{T}}\right\} I_3 - \frac{1}{2}\Sigma_\omega {\Sigma_\omega}^{\mathrm{T}}. \tag{8}$$

To show the validity of usage of the system (7) as a stochastic system model of the rotational motion, we refer to the following proposition.

Proposition 3.1. *(Yamauchi et al., 2014) Consider the Itô SDE (7). If* $e^{\widehat{\xi}\theta}(0) \in SO(3)$ *holds, then* $e^{\widehat{\xi}\theta}(t) \in SO(3)$ *for all* $t \geq 0$ *with probability 1.*

Next we consider the translational motion. Under prescribed setting, the following ordinal differential equation is first employed:

$$\dot{p} = e^{\widehat{\xi}\theta}(u_p + \gamma_v).$$

Here, we write v as u_p to clarify that the velocity is supposed to be the control input. Similarly to the rotational motion, the Wong-Zakai approximation theorem implies that the translational motion is formulated as the following Stratonovich SDE:

$$\mathrm{d}p = e^{\widehat{\xi}\theta}u_p\mathrm{d}t + e^{\widehat{\xi}\theta} \circ \mathrm{d}w_v, \tag{9}$$

where $w_v \in \mathbb{R}^3$ denotes a Wiener process satisfying $\mathbb{E}[\mathrm{d}w_v \mathrm{d}w_v^{\mathrm{T}}] = \Sigma_v \Sigma_v^{\mathrm{T}} \mathrm{d}t$. Equation (9) is converted into the following almost sure equivalent Itô SDE:

$$\mathrm{d}p = e^{\widehat{\xi}\theta}(-A_2(\Sigma_v) + u_p)\mathrm{d}t + e^{\widehat{\xi}\theta}\mathrm{d}w_v. \tag{10}$$

Here, the Wong-Zakai correction term denoted by $A_2(\Sigma_v)$ is given by

$$A_2(\Sigma_v) := \frac{1}{2}\sum_{j=1}^{r} \widehat{[\Sigma_v]_{:,j}}[\Sigma_\omega]_{:,j},$$

where $[\cdot]_{i,j}$, $[\cdot]_{:,j}$ and $[\cdot]_{i,:}$ denote the (i,j)th element, the jth column vector, and the ith row vector of a matrix, respectively. Eventually, combining Eqs. (7) and (10) with the homogeneous representation (4), we have the following stochastic system of the rigid body motion of the spacecraft based on Itô SDE:

$$\mathrm{d}g = \begin{bmatrix} \mathrm{d}e^{\widehat{\xi}\theta} & \mathrm{d}p \\ 0 & 0 \end{bmatrix} = g(-\tilde{A}(\Sigma) + \widehat{u})\mathrm{d}t + g\widehat{\mathrm{d}w}. \tag{11}$$

where $\tilde{A}(\Sigma)$, $\widehat{\mathrm{d}w}$ and $\widehat{u}$ is defined as

$$\tilde{A}(\Sigma) := \begin{bmatrix} A_1(\Sigma_\omega) & A_2(\Sigma_v) \\ 0 & 0 \end{bmatrix}, \quad \widehat{\mathrm{d}w} := \begin{bmatrix} \widehat{\mathrm{d}w_\omega} & \mathrm{d}w_v \\ 0 & 0 \end{bmatrix}, \quad \widehat{u} := \begin{bmatrix} \widehat{u_R} & u_p \\ 0 & 0 \end{bmatrix}.$$

There have been some conventional papers dealing with either the rotational motion or the translational motion as a stochastic system described by an SDE. On the contrary, the present system model (11) in this paper simultaneously deals with the both motions as an SDE on $SE(3)$. Moreover, by considering the correlation of uncertainties associated with the rotational and translational motions, the interaction between the both motions is properly described.

Next, we consider the stabilization of the rigid body motion of the spacecraft, where the target state is set to $g = I_4$. We employ the following control laws for the attitude and the translational motions:

$$\widehat{u_R} = -2k_u \tan\frac{\theta}{2}\widehat{\xi}, \qquad (-\pi < \theta < \pi), \tag{12}$$

$$u_p = -k_u {e^{\widehat{\xi}\theta}}^{\mathrm{T}} p + \frac{1}{2}\sum_{i=1}^{r} \widehat{[\Sigma_v]_{:,i}}[\Sigma_\omega]_{:,i}, \tag{13}$$

where k_u is a positive gain. Then, we define a twice differentiable positive definite function $S(g)$ as a stochastic Lyapunov-like function as

$$S(g) := \phi(e^{\widehat{\xi}\theta}) + \psi(p) = \frac{1}{2}\|I_3 - e^{\widehat{\xi}\theta}\|_F^2 + \|p\|^2 = \mathrm{tr}\left\{I_3 - e^{\widehat{\xi}\theta}\right\} + p^{\mathrm{T}}p, \tag{14}$$

where $\|\cdot\|_F$ denotes the Frobenius norm, namely, for a matrix $M \in \mathbb{R}^{n\times m}$, $\|M\|_F = \sqrt{\mathrm{tr}\left\{M^{\mathrm{T}}M\right\}}$.

First, let us consider the time evolution of $\phi(e^{\widehat{\xi}\theta})$ of the function $S(g)$ in (14). Since the infinitesimal operator $\mathcal{L}(\cdot)$ in (2) is defined for a vector argument not a matrix such as $e^{\widehat{\xi}\theta}$, it is not directly applied to the function $S(g)$. Thus, we prepare the following lemma.

Lemma 3.2. *Consider an $n \times n$ matrix X and a twice differentiable function $\bar{\Phi} : \mathbb{R}^{n\times n} \to \mathbb{R}$. Then, the infinitesimal variation of $\bar{\Phi}(X)$ is given by*

$$\begin{aligned} \mathrm{d}\bar{\Phi}(X) &= \sum_{i,j=1}^{n} \frac{\partial\bar{\Phi}}{\partial X_{ij}}\mathrm{d}X_{ij} + \frac{1}{2}\sum_{i,j,k,l=1}^{n} \frac{\partial^2\bar{\Phi}}{\partial X_{ij}\partial X_{kl}}\mathrm{d}X_{ij}\mathrm{d}X_{kl} + o(\|\mathrm{d}X\|^2) \\ &= \mathrm{tr}\left\{\left(\frac{\partial\bar{\Phi}}{\partial X}\right)^{\mathrm{T}}\mathrm{d}X\right\} + \frac{1}{2}\sum_{i,j=1}^{n}\mathrm{tr}\left\{\frac{\partial}{\partial X_{ij}}\left(\frac{\partial\bar{\Phi}}{\partial X}\right)^{\mathrm{T}}\mathrm{d}X\right\}\mathrm{d}X_{ij} + o(\|\mathrm{d}X\|^2), \end{aligned} \tag{15}$$

where $\partial/\partial X$ is defined as:

$$\frac{\partial}{\partial X} = \begin{bmatrix} \frac{\partial}{\partial X_{11}} & \cdots & \frac{\partial}{\partial X_{1n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial X_{n1}} & \cdots & \frac{\partial}{\partial X_{nn}} \end{bmatrix}.$$

Proof. The first term of the right hand side of Eq. (15) is transformed as follows:

$$\begin{aligned} \left(\frac{\partial\bar{\Phi}}{\partial X}\right)^{\mathrm{T}} \mathrm{d}X &= \begin{bmatrix} \frac{\partial\bar{\Phi}}{\partial X_{11}} & \cdots & \frac{\partial\bar{\Phi}}{\partial X_{n1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial\bar{\Phi}}{\partial X_{1n}} & \cdots & \frac{\partial\bar{\Phi}}{\partial X_{nn}} \end{bmatrix} \begin{bmatrix} \mathrm{d}X_{11} & \dots & \mathrm{d}X_{1n} \\ \vdots & \ddots & \vdots \\ \mathrm{d}X_{n1} & \dots & \mathrm{d}X_{nn} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{i=1}^{n} \frac{\partial\bar{\Phi}}{\partial X_{i1}} \mathrm{d}X_{i1} & & * \\ & \ddots & \\ * & & \sum_{i=1}^{n} \frac{\partial\bar{\Phi}}{\partial X_{in}} \mathrm{d}X_{in} \end{bmatrix}, \end{aligned} \tag{16}$$

where non-diagonal terms are omitted, since they are not used. Equation (16) leads to

$$\mathrm{tr}\left\{\left(\frac{\partial\bar{\Phi}}{\partial X}\right)^{\mathrm{T}} \mathrm{d}X\right\} = \sum_{i,j=1}^{n} \frac{\partial\bar{\Phi}}{\partial X_{ij}} \mathrm{d}X_{ij}. \tag{17}$$

The second term of the right hand side of Eq. (15) is transformed as follows:

$$\begin{aligned} \frac{\partial}{\partial X_{ij}} \left(\frac{\partial\bar{\Phi}}{\partial X}\right)^{\mathrm{T}} \mathrm{d}X &= \begin{bmatrix} \frac{\partial^2\bar{\Phi}}{\partial X_{ij}\partial X_{11}} & \cdots & \frac{\partial^2\bar{\Phi}}{\partial X_{ij}\partial X_{n1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2\bar{\Phi}}{\partial X_{ij}\partial X_{1n}} & \cdots & \frac{\partial^2\bar{\Phi}}{\partial X_{ij}\partial X_{nn}} \end{bmatrix} \begin{bmatrix} \mathrm{d}X_{11} & \dots & \mathrm{d}X_{1n} \\ \vdots & \ddots & \vdots \\ \mathrm{d}X_{n1} & \dots & \mathrm{d}X_{nn} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{k=1}^{n} \frac{\partial^2\bar{\Phi}}{\partial X_{ij}\partial X_{k1}} \mathrm{d}X_{k1} & \cdots & * \\ \vdots & \ddots & \vdots \\ * & \cdots & \sum_{k=1}^{n} \frac{\partial^2\bar{\Phi}}{\partial X_{ij}\partial X_{kn}} \mathrm{d}X_{kn} \end{bmatrix}. \end{aligned} \tag{18}$$

Then, Eq. (18) also leads to

$$\mathrm{tr}\left\{\frac{\partial}{\partial X_{ij}} \left(\frac{\partial\bar{\Phi}}{\partial X}\right)^{\mathrm{T}} \mathrm{d}X\right\} = \sum_{k,l=1}^{n} \frac{\partial^2\bar{\Phi}}{\partial X_{ij}\partial X_{kl}} \mathrm{d}X_{kl}. \tag{19}$$

It follows from Eq. (19) that

$$\sum_{i,j,k,l=1}^{n} \frac{\partial^2\bar{\Phi}}{\partial X_{ij}\partial X_{kl}} \mathrm{d}X_{ij}\mathrm{d}X_{kl} = \sum_{i,j=1}^{n} \mathrm{tr}\left\{\frac{\partial}{\partial X_{ij}} \left(\frac{\partial\bar{\Phi}}{\partial X}\right)^{\mathrm{T}} \mathrm{d}X\right\} \mathrm{d}X_{ij}. \tag{20}$$

Consequently, from Eqs. (17) and (20), the assertion of the lemma is proved. □

Then, we provide the main results.

Lemma 3.3. *Consider the stochastic system (11) with the control laws (12) and (13). The function $S(g)$ in Eq. (14) satisfies the following inequalities:*

$$\frac{1}{2}\|I_4 - g\|_F^2 \leq S(g). \tag{21}$$

$$\mathcal{L}S(g) \leq -2k_u S(g) + \operatorname{tr}\left\{\Sigma\Sigma^{\mathrm{T}}\right\}, \tag{22}$$

Proof. See Appendix A. □

According to Lemma 3.3, we are now ready to state the stability theorem.

Theorem 3.4. *Consider the stochastic system (11) with the control laws (12) and (13). Then, the solution of the system is exponentially ultimately bounded in mean square sense, and there exists a positive constant c_2 such that the following inequality holds:*

$$\mathbb{E}[\|I_4 - g\|_F^2] \leq \frac{1}{2}e^{-c_2 t} + \frac{\operatorname{tr}\left\{\Sigma\Sigma^{\mathrm{T}}\right\}}{k_u}.$$

Particularly, an ultimate upper bound c_3 defined in Definition 2.1 is explicitly given by

$$c_3 = \frac{\operatorname{tr}\left\{\Sigma\Sigma^{\mathrm{T}}\right\}}{k_u}. \tag{23}$$

Proof. The assertion is directly proved using the inequalities (21) and (22) in Lemma 3.3, and Theorem 2.2. □

4. Numerical Example

In this section, we demonstrate the validity of Theorem 3.4 through a numerical simulation using the Euler-Maruyama method (Kloeden & Platen, 1992). The parameters of the simulation are given as follows. The covariance matrix of the Wiener process is set to

$$\Sigma\Sigma^{\mathrm{T}} = \begin{bmatrix} 0.04 & 0.04 & 0.04 & 0.04 & 0.04 & 0.04 \\ 0.04 & 0.04 & 0.04 & 0.04 & 0.04 & 0.04 \\ 0.04 & 0.04 & 0.04 & 0.04 & 0.04 & 0.04 \\ 0.04 & 0.04 & 0.04 & 0.04 & 0.04 & 0.04 \\ 0.04 & 0.04 & 0.04 & 0.04 & 0.04 & 0.04 \\ 0.04 & 0.04 & 0.04 & 0.04 & 0.04 & 0.04 \end{bmatrix} \in \mathbb{R}^{6\times 6},$$

and the feedback gain and the temporal increment are $k_u = 0.5$ and $\mathrm{d}t = 1\times 10^{-3}$s, respectively. The number of samples for calculation of the expectation is set to 1000. The initial position and attitude of the body frame Σ_b are selected as $p_0 = [1.00, 1.00, 2.25]^{\mathrm{T}}$ and $\xi\theta = [0, 0, 2\pi/3]$rad, respectively. Under the control laws (12) and (13), the simulation results of the rigid body motion of the spacecraft are shown in Figures 2 to 5. Figure 2 shows the expectation of the squared error norm $\|I_4 - g\|_F^2$ and Fig. 3 shows an enlarged view of Fig. 2 from 17s to 20s. The dotted line in each figure represents the ultimate upper bound c_3 given by Eq. (23) in Theorem 3.4. Here, c_3 is calculated

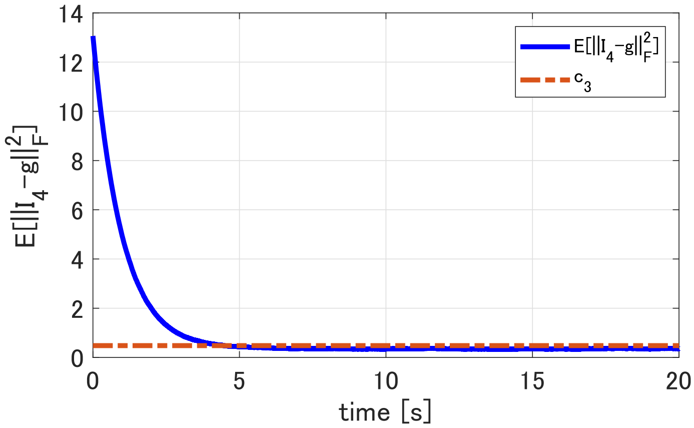

Figure 2. The expectation of $\|I_4 - g\|_F^2$ with the ultimate upper bound c_3.

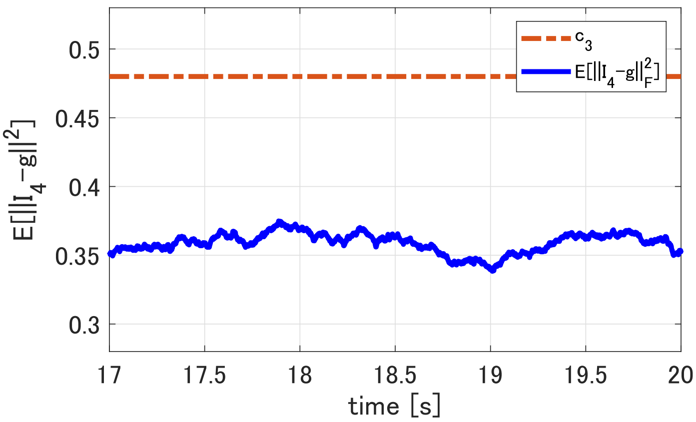

Figure 3. The expectation of $\|I_4 - g\|_F^2$ with the bound c_3 from 17s to 20s.

as $c_3 = 0.48$. Figure 4 shows a sample path of Euler angles α, β and γ, respectively. Finally, Fig. 5 shows a sample path of displacement $p = [p_x, p_y, p_z]^{\mathrm{T}}$. From those figures, the rigid body motion of the spacecraft is approaching to the desired one even under persistent noise, and moreover the expectation of the squared error norm $\|I_4 - g\|_F^2$ is eventually bounded by c_3 as proved in Theorem 3.4.

5. Conclusion

This paper has proposed a stochastic stabilizing method of the rigid body motion kinematics of a spacecraft. First, we have formulated the time evolution of the rigid body motion as a stochastic differential equation on the special Euclidean group $SE(3)$. Second, we have presented a stochastic stabilizing control law and a stochastic stability theorem based on exponentially ultimate boundedness in mean square sense. Finally, a numerical example has demonstrated the effectiveness of the proposed method.

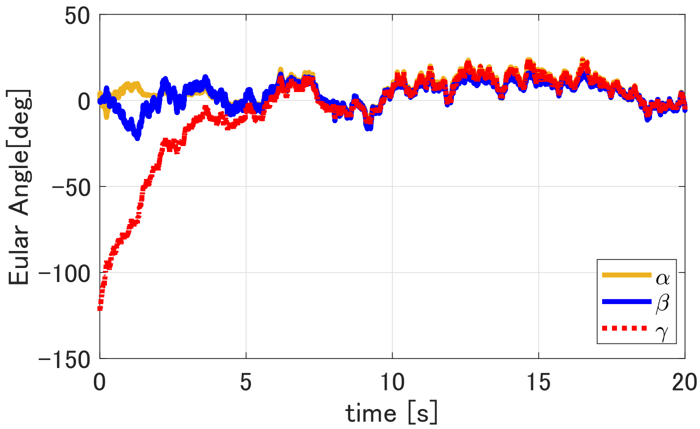

Figure 4. A sample path of Euler Angles α, β and γ.

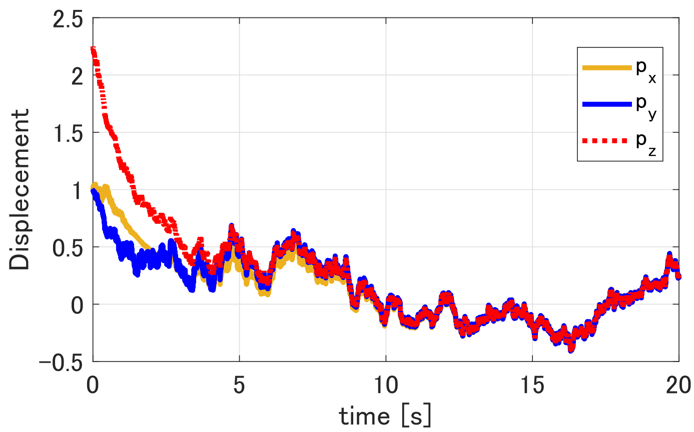

Figure 5. A sample path of displacement p_x, p_y, p_z.

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Appendix A. Proof of Lemma 3.3

First, we shall proof Eq. (21). The function $S(g)$ in Eq. (14) is rewritten as

$$S(g) = \frac{1}{2}\|I_3 - e^{\hat{\xi}\theta}\|_F^2 + \|p\|^2 = 3 - \mathrm{tr}\left\{e^{\hat{\xi}\theta}\right\} + \|p\|^2. \tag{A1}$$

Also, the following relation holds:

$$\begin{aligned}\frac{1}{2}\|I_4 - g\|_F^2 &= \frac{1}{2}\mathrm{tr}\left\{I_4 - g - g^{\mathrm{T}} + gg^{\mathrm{T}}\right\} = 2 - \mathrm{tr}\left\{g\right\} + \frac{1}{2}\mathrm{tr}\left\{gg^{\mathrm{T}}\right\} \\ &= 3 - \mathrm{tr}\left\{e^{\hat{\xi}\theta}\right\} + \|p\|^2.\end{aligned} \tag{A2}$$

Then, Eq. (21) comes from Eqs. (A1) and (A2).

Then, we shall proof Eq. (22). From Lemma 3.2, the infinitesimal variation up to the second order of $\phi(e^{\hat{\xi}\theta})$ of $S(g)$ in Eq. (14) is calculated as follows:

$$\begin{aligned}\mathrm{d}\phi(e^{\hat{\xi}\theta}) &= \mathrm{tr}\left\{\left(\frac{\partial\phi}{\partial e^{\hat{\xi}\theta}}\right)^{\mathrm{T}}\mathrm{d}e^{\hat{\xi}\theta}\right\} + \frac{1}{2}\sum_{i,j=1}^{3}\mathrm{tr}\left\{\frac{\partial}{\partial e^{\hat{\xi}\theta}{}_{ij}}\left(\frac{\partial\phi}{\partial e^{\hat{\xi}\theta}}\right)^{\mathrm{T}}\mathrm{d}e^{\hat{\xi}\theta}\right\}\left[\mathrm{d}e^{\hat{\xi}\theta}\right]_{ij} \\ &= \mathrm{tr}\left\{(-I_3)\mathrm{d}e^{\hat{\xi}\theta}\right\} + \frac{1}{2}\sum_{i,j=1}^{3}\mathrm{tr}\left\{\frac{\partial\left(-I_3\right)}{\partial e^{\hat{\xi}\theta}{}_{ij}}\mathrm{d}e^{\hat{\xi}\theta}\right\}\mathrm{d}e^{\hat{\xi}\theta}{}_{ij} \\ &= -\mathrm{tr}\left\{\mathrm{d}e^{\hat{\xi}\theta}\right\},\end{aligned} \tag{A3}$$

where we used the fact that for any differentiable matrix $X \in \mathbb{R}^{n\times n}$, the following holds (Harville, 1997):

$$\frac{\partial \mathrm{tr}\left\{X\right\}}{\partial X} = I_n.$$

By substituting the dynamics (7) with Eq. (A3), we have

$$\begin{aligned}\mathrm{d}\phi(e^{\widehat{\xi}\theta}) &= -\mathrm{tr}\left\{e^{\widehat{\xi}\theta}(-A_1(\Sigma_\omega) + \widehat{u_R})\mathrm{d}t + e^{\widehat{\xi}\theta}\widehat{\mathrm{d}w_\omega}\right\} \\ &= \mathrm{tr}\left\{e^{\widehat{\xi}\theta}A_1(\Sigma_\omega)\right\}\mathrm{d}t - \mathrm{tr}\left\{e^{\widehat{\xi}\theta}\widehat{u_R}\right\}\mathrm{d}t - \mathrm{tr}\left\{e^{\widehat{\xi}\theta}\widehat{\mathrm{d}w_\omega}\right\}.\end{aligned}$$

It follows from $\mathbb{E}[\mathrm{d}\phi] = \mathcal{L}\phi\mathrm{d}t$ in Eq. (3) and $\mathbb{E}[\widehat{\mathrm{d}w_\omega}] = 0$ that

$$\begin{aligned}\mathcal{L}\phi(e^{\widehat{\xi}\theta}) &= \mathrm{tr}\left\{e^{\widehat{\xi}\theta}A_1(\Sigma_\omega)\right\} - \mathrm{tr}\left\{e^{\widehat{\xi}\theta}\widehat{u_R}\right\} \\ &= \frac{1}{2}\mathrm{tr}\left\{e^{\widehat{\xi}\theta}\left(\mathrm{tr}\left\{\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}I_3 - \Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right)\right\} - \mathrm{tr}\left\{e^{\widehat{\xi}\theta}\widehat{u_R}\right\} \\ &= \frac{1}{2}\mathrm{tr}\left\{\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}\mathrm{tr}\left\{e^{\widehat{\xi}\theta}\right\} - \frac{1}{2}\mathrm{tr}\left\{e^{\widehat{\xi}\theta}\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\} - \mathrm{tr}\left\{e^{\widehat{\xi}\theta}\widehat{u_R}\right\} \\ &= \frac{1}{2}\mathrm{tr}\left\{\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}(1 + 2\mathrm{cos}\theta) - \frac{1}{2}\mathrm{tr}\left\{e^{\widehat{\xi}\theta}\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\} - \mathrm{tr}\left\{e^{\widehat{\xi}\theta}\widehat{u_R}\right\}. \qquad \text{(A4)}\end{aligned}$$

Here, we supplementarily explain the derivation of the last equality in Eq. (A4). Using Rodrigues's formula:

$$e^{\widehat{\xi}\theta} = I_3 + \widehat{\xi}\mathrm{sin}\theta + \widehat{\xi}^2(1 - \mathrm{cos}\theta), \qquad \text{(A5)}$$

we have

$$\mathrm{tr}\left\{e^{\widehat{\xi}\theta}\right\} = 3 + (1 - \cos\theta)\mathrm{tr}\left\{\widehat{\xi}^2\right\}.$$

Then, $\mathrm{tr}\{\widehat{\xi}^2\} = -2$ follows the fact that for any $\zeta \in \mathbb{R}^3$, $\widehat{\zeta}^2 = \zeta\zeta^{\mathrm{T}} - \|\zeta\| I_3$ holds, which leads to the last equality in Eq. (A4). Let us evaluate the term $-\mathrm{tr}\left\{e^{\widehat{\xi}\theta}\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}$ in the last equality in Eq. (A4). Let $\mathrm{sym}(e^{\widehat{\xi}\theta})$ and $\mathrm{sk}(e^{\widehat{\xi}\theta})$ be symmetric and skew-symmetric components of $e^{\widehat{\xi}\theta}$, respectively. That is, $e^{\widehat{\xi}\theta}$ is uniquely decomposed as

$$e^{\widehat{\xi}\theta} = \mathrm{sym}(e^{\widehat{\xi}\theta}) + \mathrm{sk}(e^{\widehat{\xi}\theta}) = \frac{e^{\widehat{\xi}\theta} + {e^{\widehat{\xi}\theta}}^{\mathrm{T}}}{2} + \frac{e^{\widehat{\xi}\theta} - {e^{\widehat{\xi}\theta}}^{\mathrm{T}}}{2}. \qquad \text{(A6)}$$

Using Eq. (A6), $-\mathrm{tr}\left\{e^{\widehat{\xi}\theta}\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}$ is calculated as follows:

$$-\mathrm{tr}\left\{e^{\widehat{\xi}\theta}\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\} = -\mathrm{tr}\left\{(\mathrm{sym}(e^{\widehat{\xi}\theta}) + \mathrm{sk}(e^{\widehat{\xi}\theta}))\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\} = -\mathrm{tr}\left\{\mathrm{sym}(e^{\widehat{\xi}\theta})\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}, \qquad \text{(A7)}$$

where in the last equality, the fact that for any $M = M^{\mathrm{T}} \in \mathbb{R}^{3\times 3}$ and $N = -N^{\mathrm{T}} \in$

$\mathbb{R}^{3\times 3}$, $\mathrm{tr}\{MN\}=0$ holds, is utilized. Then, Eq. (A7) is reduced to

$$\begin{aligned}-\mathrm{tr}\left\{\mathrm{sym}(e^{\widehat{\xi}\theta})\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\} &= -\mathrm{tr}\left\{\left(I_3+(1-\cos\theta)\widehat{\xi}^2\right)\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}\\ &= -\mathrm{tr}\left\{\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}-(1-\cos\theta)\mathrm{tr}\left\{\widehat{\xi}^2\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}. \end{aligned}\tag{A8}$$

According to the literature (Mori, 1988), $\mathrm{tr}\left\{\widehat{\xi}^2\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}$ is evaluated as

$$\begin{aligned}&\lambda_{\min}\left(\widehat{\xi}^2\right)\mathrm{tr}\left\{\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}\le \mathrm{tr}\left\{\widehat{\xi}^2\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\},\\ \Rightarrow\quad &-\mathrm{tr}\left\{\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}\le \mathrm{tr}\left\{\widehat{\xi}^2\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\},\end{aligned}\tag{A9}$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of the argument, and a simple calculation analytically reveals the eigenvalues of $\widehat{\xi}^2$ as 0 and -1. Hence, substitution of Eqs. (A7), (A8) and (A9) with (A4) gives the following evaluation:

$$\begin{aligned}&\mathcal{L}\phi(e^{\widehat{\xi}\theta})\\ &=\frac{1}{2}\mathrm{tr}\left\{\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}(1+2\cos\theta)-\frac{1}{2}\mathrm{tr}\left\{\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}-\frac{1}{2}(1-\cos\theta)\mathrm{tr}\left\{\widehat{\xi}^2\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}-\mathrm{tr}\left\{e^{\widehat{\xi}\theta}\widehat{u_R}\right\}\\ &\le\frac{1}{2}\mathrm{tr}\left\{\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}(1+2\cos\theta)-\frac{1}{2}\mathrm{tr}\left\{\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}+\frac{1}{2}(1-\cos\theta)\mathrm{tr}\left\{\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}-\mathrm{tr}\left\{e^{\widehat{\xi}\theta}\widehat{u_R}\right\}\\ &\le \mathrm{tr}\left\{\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}-\mathrm{tr}\left\{e^{\widehat{\xi}\theta}\widehat{u_R}\right\}.\end{aligned}\tag{A10}$$

Then, we evaluate the last term in the right hand side of the inequality (A10). By substituting the control law defined in Eq. (12) into the last term, we have

$$\begin{aligned}-\mathrm{tr}\left\{e^{\widehat{\xi}\theta}\widehat{u_R}\right\}&=2k_u\tan\frac{\theta}{2}\mathrm{tr}\left\{e^{\widehat{\xi}\theta}\widehat{\xi}\right\}=2k_u\tan\frac{\theta}{2}\mathrm{tr}\left\{\mathrm{sk}(e^{\widehat{\xi}\theta})\widehat{\xi}\right\}\\ &=2k_u\tan\frac{\theta}{2}\sin\theta\mathrm{tr}\left\{\widehat{\xi}^2\right\}=2k_u\tan\frac{\theta}{2}\sin\theta\mathrm{tr}\left\{\xi\xi^{\mathrm{T}}-\|\xi\|I_3\right\}\\ &=-4k_u\tan\frac{\theta}{2}\sin\theta=-8k_u\sin^2\frac{\theta}{2}.\end{aligned}\tag{A11}$$

Equations (A10) and (A11) lead to

$$\mathcal{L}\phi(e^{\widehat{\xi}\theta})\le -8k_u\sin^2\frac{\theta}{2}+\mathrm{tr}\left\{\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}.\tag{A12}$$

Besides, the positive function defined in (14) is rewritten as follows:

$$\phi(e^{\widehat{\xi}\theta})=3-\mathrm{tr}\left\{e^{\widehat{\xi}\theta}\right\}=3-(1+2\cos\theta)=4\sin^2\frac{\theta}{2}.\tag{A13}$$

Consequently, it follows from Eqs. (A12) and (A13) that

$$\mathcal{L}\phi(e^{\widehat{\xi}\theta})\le -2k_u\phi(e^{\widehat{\xi}\theta})+\mathrm{tr}\left\{\Sigma_\omega{\Sigma_\omega}^{\mathrm{T}}\right\}.\tag{A14}$$

Next, let us consider the infinitesimal variation of $\psi(p)$ of the positive function $S(g)$. From the definition of the infinitesimal operator $\mathcal{L}(\cdot)$ defined in (2), we have

$$\begin{aligned}\mathcal{L}\psi(p) &= \frac{\partial \psi(p)}{\partial p} e^{\widehat{\xi}\theta}\left(-A_2(\Sigma_v) + u_p\right) + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial p}\left(\frac{\partial \psi(p)}{\partial p}\right)^{\mathrm{T}} e^{\widehat{\xi}\theta}\Sigma_v {\Sigma_v}^{\mathrm{T}} {e^{\widehat{\xi}\theta}}^{\mathrm{T}}\right\} \\ &= 2p^{\mathrm{T}} e^{\widehat{\xi}\theta}(-A_2(\Sigma_v) + u_p) + \mathrm{tr}\left\{{\Sigma_v}^{\mathrm{T}}\Sigma_v\right\}. \end{aligned} \tag{A15}$$

By substituting the control law defined as (13) with Eq. (A15), we have

$$\begin{aligned}\mathcal{L}\psi(p) &= 2p^{\mathrm{T}} e^{\widehat{\xi}\theta}\left(-k_u {e^{\widehat{\xi}\theta}}^{\mathrm{T}} p\right) + \mathrm{tr}\left\{{\Sigma_v}^{\mathrm{T}}\Sigma_v\right\} \\ &= -2k_u\psi(p) + \mathrm{tr}\left\{{\Sigma_v}^{\mathrm{T}}\Sigma_v\right\}. \end{aligned} \tag{A16}$$

Equation (21) comes from Eqs. (A14) and (A16).