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Input-to-state stability of stochastic port-Hamiltonian systems using stochastic generalized canonical transformations

Satoshi Satoh[1]*

[1]*Division of Mechanical Systems and Applied Mechanics, Faculty of Engineering, Hiroshima University, 1-4-1, Kagamiyama, Higashi-Hiroshima 739-8527, Japan*

SUMMARY

As a practically important class of nonlinear stochastic systems, this paper considers stochastic port-Hamiltonian systems (SPHSs), and investigates the stochastic input-to-state stability (SISS) property of a class of SPHSs. We clarify necessary conditions for the closed-loop system of an SPHS to be SISS. Moreover, we provide a systematic construction of both the SISS controller and Lyapunov function so that the proposed necessary conditions hold. In the main results, the stochastic generalized canonical transformation (SGCT) plays a key role. The SGCT technique enables to design both coordinate transformation and feedback controller with preserving the SPHS structure of the closed-loop system. Consequently, the main theorem guarantees that the closed-loop system obtained by the proposed method is SISS against both deterministic disturbance and stochastic noise.

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KEY WORDS: Stochastic control, Nonlinear control, Robust control, Stochastic input-to-state stability, Stochastic Hamiltonian systems

1. INTRODUCTION

Stochastic input-to-state stability (SISS) is well recognized as an important concept for analysis and synthesis of nonlinear stochastic systems. SISS guarantees that the boundedness of sample paths with an arbitrary probability against essentially bounded unknown external disturbances. The notion of SISS is an extension of the deterministic input-to-state stability (ISS) concept [1], and has been improved by several researchers, e.g., [2, 3, 4, 5]. The SISS concept in [3] was developed from the concept of γ- input-to-state stability in [2]. The literature [3, 2] only considers the deterministic external disturbances. Then, the authors in [4] extend the SISS concept to deal with stochastic external disturbances as well. The aforementioned literature considers a general class of nonlinear stochastic systems, and sufficient conditions for SISS based on the SISS Lyapunov function are provided, which are useful for analysis and controller design. However, since general nonlinear

*Correspondence to: Hiroshima University, 1-4-1, Kagamiyama, Higashi-Hiroshima 739-8527, Japan. E-mail: s.satoh@ieee.org

systems are focused, concrete construction of the SISS Lyapunov function has not been provided enough so far.

As a practically important class of nonlinear stochastic systems, the author has introduced stochastic port-Hamiltonian systems (SPHSs) in [6]. SPHS is an extension of the deterministic port-Hamiltonian system (PHS) [7], and it represents electrical and mechanical systems, electromechanical systems and systems with nonholonomic constraints with uncertainties such as system and measurement noises and modeling error. Moreover, both PHSs and SPHSs have a nice property for control that the Hamiltonian of the system is a candidate of the Lyapunov function. Regarding this, the main purpose of this paper is to provide a systematic construction of the controller and Lyapunov function for endowing SPHSs with (practically) SISS property. As pointed out in the literature [8], there are few studies on robustness of PHSs against external disturbances. Particularly, this important subject has not been clarified for SPHSs at all. The key techniques in this paper are coordinate transformations, integral actions and feedback compensators derived from the stochastic generalized canonical transformations (SGCTs). A step-by-step construction of the ISS controller for PHSs has been proposed in [8], where particular coordinate transformations in [9, 8] and an integral action in [8] are respectively equipped. We extend this construction method to the case of SPHSs. There are two main differences from the literature [9, 8]. In [6], we have revealed that an extra condition due to existence of the stochastic noise is necessary to preserve the SPHS structure under a coordinate transformation compared to the case of PHS. Hence, the conventional methods preserving PHS structure in [9, 8] cannot be directly applied to SPHSs. SGCTs proposed by the author in [6] provide conditions for the coordinate and feedback transformations to preserve the SPHS structure. Regarding this, first, we newly derive some feedback compensators to solve this problem by using the SGCT technique. The proposed compensators are peculiar to existence of stochastic noise, and thus they do not appear in the PHS case. Second, although the method in [8] merely deals with matched deterministic disturbances, the proposed method can deal with both matched and unmatched stochastic noises as well. Thanks to stochastic calculus, we explicitly provide necessary conditions of the noise port of the plant system as well as some design parameters of the controller for achieving SISS property. This paper gives a step-by-step construction of both the controller and Lyapunov function so that the proposed necessary conditions hold. Consequently, it is guaranteed that the closed-loop system is (practically) SISS against both deterministic disturbance and stochastic noise.

In this paper, for a vector a, a_j denotes its j th element, and for a matrix A, $[A]_{j,k}$, $[A]_{j,:}$ and $[A]_{:,k}$ denote its (j,k) th element, j th row and k th column, respectively. $\mathrm{Sym}(A)$ represents the symmetric part of the matrix A, i.e., $\mathrm{Sym}(A) := (A + A^\top)/2$. $\|A\|, \|A\|_F$ denotes the induced norm and Frovenius norm, i.e., $\|A\| := \sqrt{\lambda_{\max}(AA^\top)}$ and $\|A\|_F := \sqrt{\mathrm{tr}\{AA^\top\}}$, respectively. $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ and $\mathrm{tr}\{\cdot\}$ represent the minimum and maximum eigenvalues, and the trace of $(\cdot)$, respectively. We use the Euclidean norm for vectors. Besides, we define $\partial F(x)/\partial x$ for a class C^2 function $F(x)$ as an n-dimensional row vector such that its i th column is given by $\partial F(x)/\partial x_i$. We also define $\partial^2 F(x)/\partial x^2$ as an $n \times n$ matrix such that its (i,j) th element is given by $\partial^2 F(x)/\partial x_j \partial x_i$. Also, I_n and $0_{i\times j}$ denote the $n \times n$ identity matrix and $i \times j$ zero matrix, respectively. We sometimes use I and 0 by dropping the subscripts for notational brevity.

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2. PRELIMINARIES

We consider a class of stochastic port-Hamiltonian systems (SPHSs) proposed in [6] with external disturbances †, which is described by the following Itô stochastic differential equation:

$$\mathrm{d}x = (J(x) - R(x)) \frac{\partial H(x)}{\partial x}^{\top} \mathrm{d}t + g(x)u\, \mathrm{d}t + k(x)v\, \mathrm{d}t + h(x)\, \mathrm{d}w \tag{1}$$

with the Hamiltonian $H(x) \in \mathbb{R}$, where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$ denote the state and the control input, and $v(t) \in \mathbb{R}^{m_v}$ represents an essentially locally bounded measurable disturbance, respectively. Here, $J(x) \in \mathbb{R}^{n \times n}$ and $R(x) \in \mathbb{R}^{n \times n}$ are skew-symmetric and symmetric positive semi-definite for all x, respectively. $w(t) \in \mathbb{R}^{m_w}$ denotes the standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is a sample space, $\mathcal{F}$ is the sigma algebra of the observable random events and $\mathcal{P}$ is a probability measure on Ω. A filtration $\{\mathcal{F}_t\}$ represents an increasing family of σ-algebras with $\mathcal{F}_t \subset \mathcal{F}, \ \forall t \geq 0$. We suppose that $\{\mathcal{F}_t\}$ is right-continuous and complete. $g(x) \in \mathbb{R}^{n \times m}$, $k(x) \in \mathbb{R}^{n \times m_v}$ and $h(x) \in \mathbb{R}^{n \times m_w}$ represent the control, disturbance and noise ports, respectively. We suppose that the Hamiltonian H is a sufficiently smooth function, and that the input u is an $\mathbb{R}^m$-valued measurable function and satisfies $E[\int_0^t \|u(s)\|^2 \, \mathrm{d}s] < \infty, \ \forall t \geq 0$ with the expectation with respect to the measure $\mathcal{P}$, denoted by $E[\cdot]$.

Particularly in the sequel, this paper concentrates on the following typical class of SPHS (1):

$$\begin{pmatrix} \mathrm{d}q \\ \mathrm{d}p \end{pmatrix} = \begin{pmatrix} 0_{m \times m} & I_m \\ -I_m & 0_{m \times m} \end{pmatrix} \begin{pmatrix} \frac{\partial H_0(q,p)}{\partial q}^{\top} \\ \frac{\partial H_0(q,p)}{\partial p}^{\top} \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0_{m \times m} \\ I_m \end{pmatrix} u \, \mathrm{d}t + \begin{pmatrix} 0_{m \times m} \\ I_m \end{pmatrix} v \, \mathrm{d}t + \begin{pmatrix} h_{11}(q,p) \, \mathrm{d}w_1 \\ h_{22}(q,p) \, \mathrm{d}w_2 \end{pmatrix} \tag{2}$$

with the Hamiltonian $H_0(q,p) = \frac{1}{2} p^{\top} M(q)^{-1} p + U_0(q)$, where $q(t), p(t) \in \mathbb{R}^m$ respectively denote the generalized coordinate and momentum, a symmetric positive definite matrix $M(q) \in \mathbb{R}^{m \times m}$ denotes the inertia matrix, and a sufficiently differentiable function $U_0(q) \in \mathbb{R}$ denotes a potential energy of the system. We define the state as $x(t) := (q(t)^{\top}, p(t)^{\top})^{\top}$ with $n = 2m$. $w_1(t) \in \mathbb{R}^{m_{w1}}$ and $w_2(t) \in \mathbb{R}^{m_{w2}}$ denote the standard Wiener processes, respectively. $h_{11}(q,p) \in \mathbb{R}^{m \times m_{w1}}$ and $h_{22}(q,p) \in \mathbb{R}^{m \times m_{w2}}$ represent the noise ports, respectively. Here, we suppose that $h_{11}(0,0) = h_{22}(0,0) = 0$ holds. $v(t) \in \mathbb{R}^m$ denotes an essentially locally bounded measurable disturbance. We merely consider matched disturbance v following the deterministic result in [8], while we can deal with both matched and unmatched stochastic noise effects by equipping w_1 and w_2. The existence and uniqueness of a strong solution to the system (2) on $t \geq 0$ is assumed. For the details of sufficient conditions for this assumption, see, e.g., [10] and [11]. In this paper, the noise ports h_{11} and h_{22} are supposed to satisfy local Lipschitz and linear growth conditions [10, 11].

In order to analyze the expected time variation of a Lyapunov function later, we equip the infinitesimal operator $\mathcal{L}(\cdot)$.

† The literature [6] considers the case where J, R, H, g and h are time-varying, while it does not deal with external disturbance effect v

Definition 1
Consider the nonlinear stochastic system written in the sense of Itô:

$$\mathrm{d}x = f(x,u,v)\,\mathrm{d}t + h(x,u,v)\,\mathrm{d}w \tag{3}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^{m_v}$ represent the state, the input and the external disturbance, respectively. The functions f and h are supposed to be locally Lipschitz in their arguments. Moreover, suppose that $f(0,0,0) = h(0,0,0) = 0$ holds. The infinitesimal generator associated with the stochastic process of the system (3) is defined as

$$\mathcal{L}(\cdot) := \frac{\partial(\cdot)}{\partial x} f + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial^2(\cdot)}{\partial x^2} h h^\top\right\}. \tag{4}$$

Then, we introduce the notation of (practically) stochastic input-to-state stability (SISS), and (practically) SISS Lyapunov function following the literature [4].

Definition 2
The system (3) is practically stochastic input-to-state stable if $\forall \epsilon > 0$, there exist a class $\mathcal{KL}$ function η, a class $\mathcal{K}$ function γ and a constant $\delta \geq 0$ such that

$$\mathcal{P}\left\{\|x(t)\| < \eta(\|x(0)\|, t) + \gamma\left(\sup_{0\leq s\leq t}\|v(s)\|_{\mathrm{ess}}\right) + \delta\right\} \geq 1 - \epsilon, \qquad \forall t \geq 0,\ \ \forall x(0) \in \mathbb{R}^n \setminus \{0\}, \tag{5}$$

where $\|v(s)\|_{\mathrm{ess}} := \inf_{\mathcal{A}\subset\Omega, \mathcal{P}(\mathcal{A})=0} \sup_{t,\Omega\setminus\mathcal{A}}\{\|v(t)\|\}$. When $\delta = 0$ in (5), the system (3) is said to be stochastic input-to-state stable.

Proposition 1
[4] Consider the system (3), and suppose that there exist a C^2 function $V(x)$, class $\mathcal{K}_\infty$ functions α_1, α_2 and α_3, a class $\mathcal{K}$ function ρ, and a constant $\bar{\delta} \geq 0$, such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \tag{6}$$

$$\mathcal{L}V \leq -\alpha_3(\|x\|) + \rho(\|v\|) + \bar{\delta}. \tag{7}$$

Then, the system is practically SISS. When, $\bar{\delta} = 0$ in (7), the system is SISS.

3. MAIN RESULTS

First, we prepare a useful tool for controller design of an SPHS. The stochastic generalized canonical transformation (SGCT) proposed in [6] is a particular pair of coordinate and feedback transformations preserving the SPHS structure. Feedback controllers preserving the deterministic PHS structure, e.g. in [12, 13], do not necessarily preserve the SPHS structure, because of the noise effects. Since the literature [6] does not consider the external disturbance v, we adapt the result of the SGCT in [6] for this paper.

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Definition 3
A set of transformations

$$\begin{aligned} \bar{x} &= \Phi(x) \\ \bar{H}(\bar{x}) &= H(x) + U(x) \mid_{x=\Phi^{-1}(\bar{x})} \\ \bar{u} &= u + \beta(x) \mid_{x=\Phi^{-1}(\bar{x})} \end{aligned} \tag{8}$$

is said to be a stochastic generalized canonical transformation (SGCT) for an SPHS (1) with the external disturbance, if it transforms the system into another one which is also of the form (1) with the Hamiltonian $\bar{H}(\bar{x})$. Here, $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is a coordinate transformation, $U : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function and $\beta : \mathbb{R}^n \to \mathbb{R}^m$ is a measurable function, respectively.

Theorem 1
Consider the system in (1). A set of transformations defined by (8) with the functions $\Phi(x)$, $U(x)$ and $\beta(x)$ yields an SGCT if and only if there exist a skew-symmetric matrix $P(x)$, a symmetric matrix $Q(x)$ such that $R(x) + Q(x)$ is positive semi-definite, and the functions $\Phi(x)$, $U(x)$ and $\beta(x)$ satisfy

$$\frac{1}{2}\mathrm{tr}\left\{\frac{\partial^2 \Phi_i}{\partial x^2}\, hh^\top\right\} = \frac{\partial \Phi_i}{\partial x}\left[(J-R)\frac{\partial U}{\partial x}^\top + g\beta + (P-Q)\frac{\partial (H+U)}{\partial x}^\top\right],\ \ (i = 1, 2, \dots, n). \tag{9}$$

Proof
The claim follows from the same line as the proof of Theorem 1 in [6]. □

By following the literature [9], we first convert the system (2) into another one, where the inertia matrix becomes independent of the generalized coordinate, namely a constant matrix, to let the controller design easy. However, the original coordinate transformation used in [9] for the deterministic PHS is not directly applied, since it does not preserve the SPHS structure due to the stochastic noise effect. Thus, we extend the transformation to an SPHS by using the SGCT provided in Theorem 1.

Proposition 2
Consider the system (2) and the following coordinate transformation:

$$\bar{x} := \begin{pmatrix} \bar{q} \\ \bar{p} \end{pmatrix} = \begin{pmatrix} \Lambda q \\ T(q)\Lambda^{-\top} p \end{pmatrix} =: \Phi(x), \tag{10}$$

where $\Lambda \in \mathbb{R}^{m\times m}$ is a constant nonsingular matrix and $T(q) \in \mathbb{R}^{m\times m}$ is defined as the symmetric square root matrix of $\Lambda M(q)^{-1}\Lambda^\top$, namely as the positive definite matrix satisfying for all $q \in \mathbb{R}^m$, $T(q) = T(q)^\top$ and

$$T(q)^2 = \Lambda M(q)^{-1}\Lambda^\top. \tag{11}$$

Here, suppose that all the element of $T(q)$ is twice differentiable with respect to q. Moreover, consider the following feedback input:

$$u = \bar{u} - \frac{\partial (U(q) - U_0(q))}{\partial q}^\top - \Lambda^\top T(q)^{-1} \begin{pmatrix} \sum_{j,k,l=1}^{m} \frac{1}{2} \frac{\partial^2 [T\Lambda^{-\top}]_{1,j}}{\partial q_l \partial q_k} p_j [h_{11} h_{11}^\top]_{l,k} \\ \vdots \\ \sum_{j,k,l=1}^{m} \frac{1}{2} \frac{\partial^2 [T\Lambda^{-\top}]_{m,j}}{\partial q_l \partial q_k} p_j [h_{11} h_{11}^\top]_{l,k} \end{pmatrix}, \tag{12}$$

where $\bar{u} \in \mathbb{R}^m$ denotes a new control input and $U(q) \in \mathbb{R}$ denotes a sufficiently differentiable positive definite function. Then, the system (2) is converted into the following another SPHS of the form (1):

$$\begin{pmatrix} \mathrm{d}\bar{q} \\ \mathrm{d}\bar{p} \end{pmatrix} = \begin{pmatrix} 0_{m\times m} & T(\Lambda^{-1}\bar{q}) \\ -T(\Lambda^{-1}\bar{q}) & J_2(\bar{q},\bar{p}) \end{pmatrix} \begin{pmatrix} \frac{\partial \bar{H}(\bar{q},\bar{p})}{\partial \bar{q}}^\top \\ \frac{\partial \bar{H}(\bar{q},\bar{p})}{\partial \bar{p}}^\top \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0_{m\times m} \\ I_m \end{pmatrix} \hat{u}\, \mathrm{d}t + \begin{pmatrix} 0_{m\times m} \\ \bar{k}_2(\bar{q}) \end{pmatrix} v\, \mathrm{d}t$$
$$+ \begin{pmatrix} \bar{h}_{11}(\bar{q},\bar{p}) & 0_{m\times m_{w2}} \\ \bar{h}_{21}(\bar{q},\bar{p}) & \bar{h}_{22}(\bar{q},\bar{p}) \end{pmatrix} \begin{pmatrix} \mathrm{d}w_1 \\ \mathrm{d}w_2 \end{pmatrix}. \tag{13}$$

with the new Hamiltonian

$$\bar{H}(\bar{q},\bar{p}) = \frac{1}{2}\bar{p}^\top \bar{p} + \bar{U}(\bar{q}), \tag{14}$$

where $\bar{U}(\bar{q}) := U(\Lambda^{-1}\bar{q})$. Also, we define

$$\hat{u} := T\Lambda^{-\top}\bar{u} \tag{15}$$

$$J_2(\bar{q},\bar{p}) := \frac{\partial (T\Lambda^{-\top}p)}{\partial q}\Lambda^{-1}T - T\Lambda^{-\top}\frac{\partial (T\Lambda^{-\top}p)}{\partial q}^\top \Bigg|_{x=\Phi^{-1}(\bar{x})}$$
$$\bar{k}_2(\bar{q}) := T\Lambda^{-\top}\Big|_{x=\Phi^{-1}(\bar{x})}$$
$$\bar{h}_{11}(\bar{q},\bar{p}) := \Lambda h_{11}\Big|_{x=\Phi^{-1}(\bar{x})}$$
$$\bar{h}_{21}(\bar{q},\bar{p}) := \frac{\partial (T\Lambda^{-\top}p)}{\partial q} h_{11}\Bigg|_{x=\Phi^{-1}(\bar{x})}$$
$$\bar{h}_{22}(\bar{q},\bar{p}) := T\Lambda^{-\top}h_{22}\Bigg|_{x=\Phi^{-1}(\bar{x})}. \tag{16}$$

Proof
See Appendix A. □

Remark 1
The second term in Eq. (12) is equipped to change the inherent potential energy of the system U_0 into an assignable one U as a design parameter. The last term plays a fundamental role in preserving the SPHS structure. Indeed, only the coordinate transformation (10) preserves the deterministic PHS structure, while the stochastic case additionally requires the last term in Eq. (12) in order to compensate the stochastic noise effect. Thus, Proposition 2 is not a trivial extension of the deterministic results, e.g., [9, 8, 14].

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Next, we design a controller to endow the resultant system in (13) with disturbance attenuation property. To do this, we first add an integrator dynamics to the system, which is practically useful to eliminate the steady-state error. Then, the coordinate transformation and feedback controller are designed by the SGCT to add the proper damping effect and to preserve the SPHS structure. These procedures for the SPHSs are modification and extension of the results for the deterministic PHSs. Implementation of the integrator dynamics is based on [15, 8, 14]. However, we add an extra design parameter to the coordinate transformation proposed in [8]. This modification is fundamental to dealing with matched and unmatched noise effects. As the same reason in Remark 1, since the feedback controller proposed in [8] cannot preserve the SPHS structure, we also extend it by using the SGCT technique.

Proposition 3
Consider the system (13), and the following integrator dynamics

$$\mathrm{d}\bar{r} = (T + R_3 R_1)\frac{\partial \bar{U}}{\partial \bar{q}}^{\top}\,\mathrm{d}t + R_3 \bar{p}\,\mathrm{d}t \tag{17}$$

and feedback controller

$$\begin{aligned}\hat{u} = &-\left(R_1 \frac{\partial^2 \bar{U}}{\partial \bar{q}^2} T + J_2 + R_2 + R_3\right)\bar{p} - (R_2 + R_3)\bar{r} - (T + (R_2 + R_3)R_1)\frac{\partial \bar{U}}{\partial \bar{q}}^{\top} \\ &- \begin{pmatrix} \sum_{j,k,l=1}^{m} \frac{1}{2}[R_1]_{1,j} \frac{\partial^3 \bar{U}}{\partial \bar{q}_l \partial \bar{q}_k \partial \bar{q}_j}[\bar{h}_{11}\bar{h}_{11}^{\top}]_{l,k} \\ \vdots \\ \sum_{j,k,l=1}^{m} \frac{1}{2}[R_1]_{m,j} \frac{\partial^3 \bar{U}}{\partial \bar{q}_l \partial \bar{q}_k \partial \bar{q}_j}[\bar{h}_{11}\bar{h}_{11}^{\top}]_{l,k} \end{pmatrix},\end{aligned} \tag{18}$$

where $R_1, R_2, R_3 \in \mathbb{R}^{m\times m}$ denote symmetric positive definite matrices, respectively. Particularly, R_1 is such that the symmetric part of TR_1, namely, $\mathrm{Sym}(TR_1) = (TR_1 + R_1^{\top}T)/2$ becomes positive definite. Then, the closed-loop system described in the new coordinate

$$z := \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \bar{q} \\ \bar{p} + R_1 \frac{\partial \bar{U}(\bar{q})}{\partial \bar{q}}^{\top} + \bar{r} \\ \bar{r} \end{pmatrix} =: \Psi(\bar{q}, \bar{p}, \bar{r}) \tag{19}$$

is converted into another SPHS of the form (1):

$$\begin{aligned}\begin{pmatrix} \mathrm{d}z_1 \\ \mathrm{d}z_2 \\ \mathrm{d}z_3 \end{pmatrix} = &\begin{pmatrix} -TR_1 & T & -T \\ -T & -R_2 & -R_3 \\ T & R_3 & -R_3 \end{pmatrix} \frac{\partial H_z(z)}{\partial z}^{\top}\mathrm{d}t + \begin{pmatrix} 0_{m\times m} \\ k_{z2}(z_1) \\ 0_{m\times m} \end{pmatrix} v\,\mathrm{d}t \\ &+ \begin{pmatrix} h_{z11}(z) & 0 \\ h_{z21}(z) & h_{z22}(z) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}w_1 \\ \mathrm{d}w_2 \end{pmatrix}\end{aligned} \tag{20}$$

with the Hamiltonian

$$H_z(z) = \bar{U}(z_1) + \frac{1}{2} z_2^{\top} z_2 + \frac{1}{2} z_3^{\top} z_3, \tag{21}$$

where

$$
\begin{aligned}
k_{z2}(z_1) &:= \bar{k}_2(\bar{q})\Big|_{\bar{q}=z_1}\\
h_{z11}(z) &:= \bar{h}_{11}(\bar{q},\bar{p})\Big|_{\substack{\bar{q}=z_1\\ \bar{p}=z_2-\frac{\partial \bar{U}}{\partial z_1}-z_3}}\\
h_{z21}(z) &:= R_1\frac{\partial^2\bar{U}}{\partial \bar{q}^2}\bar{h}_{11}(\bar{q},\bar{p})+\bar{h}_{21}(\bar{q},\bar{p})\Big|_{\substack{\bar{q}=z_1\\ \bar{p}=z_2-\frac{\partial \bar{U}}{\partial z_1}-z_3}}\\
h_{z22}(z) &:= \bar{h}_{22}(\bar{q},\bar{p})\Big|_{\substack{\bar{q}=z_1\\ \bar{p}=z_2-\frac{\partial \bar{U}}{\partial z_1}-z_3}}.
\end{aligned}
\tag{22}
$$

Proof
See Appendix B. □

Remark 2
We clarify the significance of the proposed transformations (18) and (19) with comparisons with the deterministic results in [8, 14]. The first aspect is the damping structure. The extra design parameter, which does not appear in [8], is R_1. By choosing R_1, R_2 and R_3 appropriately, the damping effect is arbitrarily assignable to the dynamics of all the variables in the resultant system (20). R_1 enables one to attenuate unmatched noise presented in the dynamics of z_1 in Eq. (20). On the contrary, the literature [14] has equipped a design parameter to the dynamics of the generalized coordinate, which corresponds to the dynamics of z_1. However, it is merely a scalar parameter. Moreover, the resultant system in [14] does not have damping term in its integrator dynamics corresponding to the dynamics of z_3. Thus, this paper with matrices R_1, R_2 and R_3 enjoys much degrees of freedom in designing controller. The second aspect is compensation of the noise effect. The last term in the feedback controller (18) is unique compared to the literature [8, 14], and is fundamental to preserve the SPHS structure in the presence of noise.

Now, we prove (practically) SISS property of the system (13) against both deterministic disturbance and stochastic noise. Before showing main theorem, we summarize several assumptions.

Assumption 1
The inertia matrix $M(q) \in \mathbb{R}^{m\times m}$ is symmetric positive definite for all $q \in \mathbb{R}^m$. Moreover, for the symmetric square root matrix of $M(q)$, suppose that there exist positive constants $\underline{K}_{M^{\frac{1}{2}}}, \overline{K}_{M^{\frac{1}{2}}}$ satisfying, for all $q \in \mathbb{R}^m$,

$$\underline{K}_{M^{\frac{1}{2}}} \le \lambda_{\min}(M(q)^{\frac{1}{2}}) \le \lambda_{\max}(M(q)^{\frac{1}{2}}) \le \overline{K}_{M^{\frac{1}{2}}}.$$

Assumption 2
Regarding to the noise ports h_{11} and h_{22}, there exist positive constants L_{q11}, L_{p11}, δ_{11}, L_{q22}, L_{p22} and δ_{22} satisfying, for all $q, p \in \mathbb{R}^m$,

$$
\begin{aligned}
\|h_{11}(q,p)\|_F^2 &\le L_{q11}\|q\|^2 + L_{p11}\|p\|^2 + \delta_{11}^2\\
\|h_{22}(q,p)\|_F^2 &\le L_{q22}\|q\|^2 + L_{p22}\|p\|^2 + \delta_{22}^2.
\end{aligned}
$$

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Note that those conditions follow from the linear growth conditions on h_{11} and h_{22}.

Assumption 3

Assignable potential energy $U(q) \in \mathbb{R}$ is a C^3 function such that the composite function $\bar{U}(\bar{q}) := U(\Lambda^{-1}\bar{q})$ becomes positive definite, and there exist positive constants $\underline{K}_{U1}, \overline{K}_{U1}$ and $\overline{K}_{U2}$ satisfying, for all $\bar{q} \in \mathbb{R}^m$,

$$\underline{K}_{U1}\|\bar{q}\| \leq \left\| \frac{\partial \bar{U}(\bar{q})}{\partial \bar{q}}^\top \right\| \leq \overline{K}_{U1}\|\bar{q}\| \tag{23}$$

$$\operatorname{tr}\left\{ \frac{\partial^2 \bar{U}(\bar{q})}{\partial \bar{q}^2} \right\} \leq \overline{K}_{U2}. \tag{24}$$

Assumption 4

There exist positive constants L_{q21}, L_{p21} and δ_{21} satisfying, for all $q, p \in \mathbb{R}^m$,

$$\left\| \frac{\partial (T\Lambda^{-\top}p)}{\partial q} h_{11}(q,p) \right\|_F^2 \leq L_{q21}\|q\|^2 + L_{p21}\|p\|^2 + \delta_{21}^2.$$

Lemma 1

Consider the system (20), and suppose that Assumptions 1 to 4 hold. Then, there exist positive constants L_{z1}, L_{z2}, L_{z3} and δ_z satisfying, for all z,

$$\frac{1}{2}\operatorname{tr}\left\{ \frac{\partial^2 H_z}{\partial z^2} h_z h_z^\top \right\} \leq L_{z1}\|z_1\|^2 + L_{z2}\|z_2\|^2 + L_{z3}\|z_3\|^2 + \delta_z^2,$$

where L_{z1}, L_{z2}, L_{z3} and δ_z are given by

$$\begin{aligned} L_{z1} = {} & \frac{\overline{K}_{U2}\|\Lambda\|_F^2}{2}(1 + 2\overline{K}_{U2}\|R_1\|_F^2)\left(\frac{L_{q11}}{\lambda_{\min}(\Lambda)^2} + 3(\overline{K}_{M^{\frac{1}{2}}})^2 \lambda_{\max}(R_1)^2 \overline{K}_{U1}{}^2 L_{p11} \right) + \frac{L_{q21}}{\lambda_{\min}(\Lambda)^2} \\ & + 3(\overline{K}_{M^{\frac{1}{2}}})^2 \lambda_{\max}(R_1)^2 \overline{K}_{U1}{}^2 L_{p21} + \frac{m}{2(\underline{K}_{M^{\frac{1}{2}}})^2}\left(\frac{L_{q22}}{\lambda_{\min}(\Lambda)^2} + 3(\overline{K}_{M^{\frac{1}{2}}})^2 \lambda_{\max}(R_1)^2 \overline{K}_{U1}{}^2 L_{p22} \right), \end{aligned} \tag{25}$$

$$L_{z2} = L_{z3} = 3(\overline{K}_{M^{\frac{1}{2}}})^2 \left(\frac{\overline{K}_{U2}\|\Lambda\|_F^2}{2}(1 + 2\overline{K}_{U2}\|R_1\|_F^2)L_{p11} + L_{p21} + \frac{mL_{p22}}{2(\underline{K}_{M^{\frac{1}{2}}})^2} \right), \tag{26}$$

$$\delta_z^2 = \frac{\overline{K}_{U2}\|\Lambda\|_F^2}{2}(1 + 2\overline{K}_{U2}\|R_1\|_F^2)\delta_{11}^2 + \delta_{21}^2 + \frac{m\delta_{22}^2}{2(\underline{K}_{M^{\frac{1}{2}}})^2}. \tag{27}$$

Proof

See Appendix C. □

Theorem 2

Consider the system (20), and suppose that Assumptions 1 to 4 hold. Suppose that the following

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conditions hold:

$$\lambda_{\min}(\mathrm{Sym}(TR_1))\underline{K}_{U1}{}^2 > L_{z1}, \tag{28}$$

$$\lambda_{\min}(R_2) > \frac{1}{2(\underline{K}_{M^{\frac{1}{2}}})^2} + L_{z2}, \tag{29}$$

$$\lambda_{\min}(R_3) > L_{z3}. \tag{30}$$

L_{z1}, L_{z2} and L_{z3} are respectively given by Eqs. (25) and (26).

Then, the system (20) becomes practically SISS with respect to v with practically SISS Lyapunov function $H_z(z)$ in Eq. (21).

Moreover, if

$$\delta_{11} = \delta_{22} = \delta_{12} = 0 \tag{31}$$

holds in Assumptions 2 and 4, then the system (20) becomes SISS with respect to v with SISS Lyapunov function $H_z(z)$ in Eq. (21).

Proof

We calculate $\mathcal{L}H_z$ along the system (20) as

$$\begin{aligned}\mathcal{L}H_z &= -\frac{\partial H_z}{\partial z}\mathrm{diag}\{TR_1, R_2, R_3\}\frac{\partial H_z}{\partial z}^\top + \frac{\partial H_z}{\partial z_2}k_{z2}v + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial^2 H_z}{\partial z^2}h_z h_z^\top\right\} \\ &= -\frac{\partial \bar{U}}{\partial z_1}\mathrm{Sym}(TR_1)\frac{\partial \bar{U}}{\partial z_1}^\top - z_2^\top R_2 z_2 - z_3^\top R_3 z_3 + z_2^\top T\Lambda^{-\top}v + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial^2 H_z}{\partial z^2}h_z h_z^\top\right\}. \end{aligned} \tag{32}$$

We further evaluate the right hand side of Eq. (32). By equipping Lemma 1 and Assumptions 1 and 3, we have

$$\begin{aligned}\mathcal{L}H_z \leq &-\lambda_{\min}(\mathrm{Sym}(TR_1))\underline{K}_{U1}{}^2\|z_1\|^2 - \lambda_{\min}(R_2)\|z_2\|^2 - \lambda_{\min}(R_3)\|z_3\|^2 + \frac{1}{2}\|\Lambda^{-1}Tz_2\|^2 \\ &+ \frac{1}{2}\|v\|^2 + L_{z1}\|z_1\|^2 + L_{z2}\|z_2\|^2 + L_{z3}\|z_3\|^2 + \delta_z^2 \\ \leq &-\left(\lambda_{\min}(\mathrm{Sym}(TR_1))\underline{K}_{U1}{}^2 - L_{z1}\right)\|z_1\|^2 - \left(\lambda_{\min}(R_2) - \frac{1}{2(\underline{K}_{M^{\frac{1}{2}}})^2} - L_{z2}\right)\|z_2\|^2 \\ &- \left(\lambda_{\min}(R_3) - L_{z3}\right)\|z_3\|^2 + \frac{1}{2}\|v\|^2 + \delta_z^2, \end{aligned} \tag{33}$$

where the last inequality utilizes the following relation:

$$\|\Lambda^{-1}T\|^2 = \lambda_{\max}(\Lambda^{-1}TT\Lambda^{-1}) = \lambda_{\max}(M^{-1}) = \lambda_{\max}(M^{-\frac{1}{2}})^2 = \lambda_{\min}(M^{\frac{1}{2}})^{-2}.$$

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Then, the first assertion of the theorem follows from the conditions (28), (29) and (30), and Proposition 1. Moreover, the condition (31) implies $\delta_z = 0$. Thus, the estimation (33) is reduced to

$$\begin{aligned}\mathcal{L}H_z \leq &-\left(\lambda_{\min}(\mathrm{Sym}(TR_1))\underline{K}_{U1}{}^2 - L_{z1}\right)\|z_1\|^2 - \left(\lambda_{\min}(R_2) - \frac{1}{2(\underline{K}_{M^{\frac{1}{2}}})^2} - L_{z2}\right)\|z_2\|^2 \\ &- \left(\lambda_{\min}(R_3) - L_{z3}\right)\|z_3\|^2 + \frac{1}{2}\|v\|^2. \end{aligned} \tag{34}$$

The other assertion of the theorem follows from Eq. (34) with the conditions (28), (29) and (30), and Proposition 1. □

As the deterministic ISS result that ISS property is invariant under the coordinate transformation [16, 17], SISS property is also invariant under the coordinate transformation. Thus, the conclusion of Theorem 2, which shows SISS property of the resultant system (20), also implies SISS property of the original plant system (2) with the proposed controller. The following corollary explicitly claims this fact:

Corollary 1

Consider the system (2). Suppose that all the conditions of Theorem 2 hold.

Define the extended state $\chi \in \mathbb{R}^{3m}$ as

$$\chi := \begin{pmatrix} x \\ \bar{r} \end{pmatrix} = \begin{pmatrix} q \\ p \\ \bar{r} \end{pmatrix},$$

which represents the state of the system (2) with that of the integrator dynamics (17). We also define the coordinate transformation from χ to z as

$$z = \Pi(\chi) := \Psi \circ \begin{pmatrix} \Phi(x) \\ \bar{r} \end{pmatrix}, \tag{35}$$

where the coordinate transformations Φ and Ψ are given in Eqs. (10) and (19), respectively.

Then, the proposed controller, namely

$$\begin{aligned} u = &-\frac{\partial(U-U_0)}{\partial q}^\top - \Lambda^\top T^{-1}\begin{pmatrix} \sum_{j,k,l=1}^m \frac{1}{2}\frac{\partial^2[T\Lambda^{-\top}]_{1,j}}{\partial q_l \partial q_k} p_j [h_{11}h_{11}^\top]_{l,k} \\ \vdots \\ \sum_{j,k,l=1}^m \frac{1}{2}\frac{\partial^2[T\Lambda^{-\top}]_{m,j}}{\partial q_l \partial q_k} p_j [h_{11}h_{11}^\top]_{l,k} \end{pmatrix} \\ &- \Lambda^\top T^{-1}\left(R_1\Lambda^{-\top}\frac{\partial^2 U}{\partial q^2}\Lambda^{-1}T + J_2 + R_2 + R_3\right)T\Lambda^{-\top}p - \Lambda^\top T^{-1}(R_2+R_3)\bar{r} \\ &- \Lambda^\top T^{-1}(T + (R_2+R_3)R_1)\Lambda^{-\top}\frac{\partial U}{\partial q}^\top \\ &- \Lambda^\top T^{-1}\begin{pmatrix} \sum_{a,b,c,j,k,l=1}^m \frac{1}{2}[R_1]_{1,j}\frac{\partial^3 U}{\partial q_a \partial q_b \partial q_c}[\Lambda^{-1}]_{a,l}[\Lambda^{-1}]_{b,k}[\Lambda^{-1}]_{c,j}[\bar{h}_{11}\bar{h}_{11}^\top]_{l,k} \\ \vdots \\ \sum_{a,b,c,j,k,l=1}^m \frac{1}{2}[R_1]_{m,j}\frac{\partial^3 U}{\partial q_a \partial q_b \partial q_c}[\Lambda^{-1}]_{a,l}[\Lambda^{-1}]_{b,k}[\Lambda^{-1}]_{c,j}[\bar{h}_{11}\bar{h}_{11}^\top]_{l,k} \end{pmatrix} \end{aligned} \tag{36}$$

renders the system (2) with the integrator dynamics (17) practically SISS with respect to v with practically SISS Lyapunov function:

$$\begin{aligned} H_\chi(\chi) &:= H_z(z)|_{z=\Pi^{-1}(\chi)} \\ &= U(q) + \frac{1}{2}\left(T\Lambda^{-\top}p + R_1\Lambda^{-\top}\frac{\partial U}{\partial q}^\top + \bar{r}\right)^\top \left(T\Lambda^{-\top}p + R_1\Lambda^{-\top}\frac{\partial U}{\partial q}^\top + \bar{r}\right) + \frac{1}{2}\bar{r}^\top\bar{r}. \end{aligned} \tag{37}$$

Moreover, if the condition (31) holds, then the system becomes SISS with respect to v with SISS Lyapunov function $H_\chi(\chi)$ in Eq. (37).

Proof

Since the proposed controller consists of Eqs. (12), (15) and (18), it is given by Eq. (36) in the original x-coordinate using the coordinate transformations (10) and (19). Here, the following relations are used:

$$\begin{aligned} \frac{\partial \bar{U}(\bar{q})}{\partial \bar{q}} &= \frac{\partial U(q)}{\partial q}\frac{\partial q}{\partial \bar{q}} = \frac{\partial U(q)}{\partial q}\Lambda^{-1}, \\ \frac{\partial^2 \bar{U}(\bar{q})}{\partial \bar{q}^2} &= \frac{\partial}{\partial q}\left(\Lambda^{-\top}\frac{\partial U(q)}{\partial q}^\top\right)\frac{\partial q}{\partial \bar{q}} = \Lambda^{-\top}\frac{\partial^2 U(q)}{\partial q^2}\Lambda^{-1}, \\ \frac{\partial^3 \bar{U}(\bar{q})}{\partial \bar{q}_l \partial \bar{q}_k \partial \bar{q}_j} &= \sum_{a,b,c=1}^{m} \frac{\partial^3 U(q)}{\partial q_a \partial q_b \partial q_c}[\Lambda^{-1}]_{a,l}[\Lambda^{-1}]_{b,k}[\Lambda^{-1}]_{c,j}. \end{aligned}$$

From the conditions of Theorem 2 with Eq. (33), $\mathcal{L}H_z$ along the system (20) can be described as

$$\mathcal{L}H_z \le -\alpha_z(\|z\|) + \rho_z(\|v\|) + \delta_z^2 \tag{38}$$

with a class $\mathcal{K}_\infty$ function α_z and a class $\mathcal{K}$ function ρ_z, which are determined by Eq. (33). Since Π in (35) is a coordinate transformation, Lemma 1 in [17] (see, also [16]) guarantees that there exist class $\mathcal{K}_\infty$ functions α_1^Π and α_2^Π such that

$$\alpha_1^\Pi(\|\chi\|) \le \|\Pi(\chi)\| \le \alpha_1^\Pi(\|\chi\|), \quad \forall \chi \in \mathbb{R}^{3m}. \tag{39}$$

From Eqs. (38) and (39) and the fact that $\mathcal{L}H_z$ is equivalent to $\mathcal{L}H_\chi$ along the system (2) with the integrator dynamics (17), we have

$$\begin{aligned} \mathcal{L}H_\chi &\le -\alpha_z(\|\Pi(\chi)\|) + \rho_z(\|v\|) + \delta_z^2 \\ &\le -\alpha_z \circ \alpha_1^\Pi(\|\chi\|) + \rho_z(\|v\|) + \delta_z^2. \end{aligned} \tag{40}$$

Positive definiteness of $H_\chi(\chi)$ in (37) with respect to χ and Proposition 1 with Eq. (40) prove the assertion.

□

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Table I. Physical parameters

m_i	Mass of the ith link	[kg]
l_i	Length of the ith link	[m]
l_{ci}	Length to the center of gravity	[m]
$\bar{I}_i$	Inertia of the ith link	[kg m^2]
g	Gravity acceleration	$[\mathrm{m/s^2}]$

4. NUMERICAL EXAMPLES

In this section, we show some numerical examples. We design SISS controller based on Theorem 2. The literature [8] considers a prismatic robot model in their numerical simulations. However, as the authors in [8] pointed out, this robot does not satisfy one of the assumptions in their ISS controller design. Thus, we consider another robot model, which satisfies all the necessary assumptions. Here, we consider a two-link robot manipulator depicted in Fig. 1 moving on a vertical plane in the presence of noise. The joint angles of the first and the second links are denoted by θ_1 and θ_2, and the control torques are denoted by u_1 and u_2, respectively. The physical parameters of this robot are summarized in Table I. The dynamics of the robot is described as an SPHS of the form (2), where

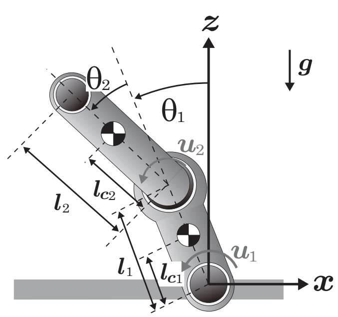

Figure 1. Two-link robot manipulator

$q := (\theta_1, \theta_2)^\top \in \mathbb{R}^2$ and $p = M(q)\dot{q} \in \mathbb{R}^2$ with the inertia matrix

$$M(q) = \begin{pmatrix} \mu_1 + 2\mu_2 \cos(q_2) & \mu_3 + \mu_2 \cos(q_2) \\ \mu_3 + \mu_2 \cos(q_2) & \mu_3 \end{pmatrix},$$

where

$$\begin{aligned} \mu_1 &:= m_1 l_{c1}^2 + m_2(l_1^2 + l_{c2}^2) + \bar{I}_1 + \bar{I}_2, \\ \mu_2 &:= m_2 l_1 l_{c2}, \\ \mu_3 &:= m_2 l_{c2}^2 + \bar{I}_2. \end{aligned} \tag{41}$$

The potential energy of the system is given by $U_0(q) = (m_1 l_{c1} + m_2 l_1) g \cos(q_1) + m_2 l_{c2} g \cos(q_1 + q_2)$.

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In order to easily obtain the analytical representation of the symmetric square root matrix of $M(q)$, we impose the following assumption:

Assumption 5

The parameters of the inertia matrix $M(q)$ in Eq. (41) satisfy

$$\mu_1 = 3\mu_3,$$
$$\mu_2 < \mu_3.$$

Note that Assumption 5 is not essential at all. Unless the analytical representation is available, the symmetric square root matrix can be numerically calculated. Under Assumption 5, the eigenvalues of $M(q)$ are given by

$$\begin{aligned} \lambda_1(M) &:= (2+\sqrt{2})\mu_3 + (1+\sqrt{2})\mu_2\cos(q_2) \\ \lambda_2(M) &:= (2-\sqrt{2})\mu_3 + (1-\sqrt{2})\mu_2\cos(q_2). \end{aligned} \tag{42}$$

Thus, the symmetric square root matrices $M(q)^{1/2}$ and $M(q)^{-1/2}$ are analytically given by

$$\begin{aligned} M(q)^{\frac{1}{2}} &= \begin{pmatrix} \frac{2+\sqrt{2}}{4}\sqrt{\lambda_1} + \frac{2-\sqrt{2}}{4}\sqrt{\lambda_2} & \frac{\sqrt{2}}{4}\sqrt{\lambda_1} - \frac{\sqrt{2}}{4}\sqrt{\lambda_2} \\ \frac{\sqrt{2}}{4}\sqrt{\lambda_1} - \frac{\sqrt{2}}{4}\sqrt{\lambda_2} & \frac{2-\sqrt{2}}{4}\sqrt{\lambda_1} + \frac{2+\sqrt{2}}{4}\sqrt{\lambda_2} \end{pmatrix}, \\ M(q)^{-\frac{1}{2}} &= \begin{pmatrix} \frac{2+\sqrt{2}}{4\sqrt{\lambda_1}} + \frac{2-\sqrt{2}}{4\sqrt{\lambda_2}} & \frac{\sqrt{2}}{4\sqrt{\lambda_1}} - \frac{\sqrt{2}}{4\sqrt{\lambda_2}} \\ \frac{\sqrt{2}}{4\sqrt{\lambda_1}} - \frac{\sqrt{2}}{4\sqrt{\lambda_2}} & \frac{2-\sqrt{2}}{4\sqrt{\lambda_1}} + \frac{2+\sqrt{2}}{4\sqrt{\lambda_2}} \end{pmatrix}. \end{aligned} \tag{43}$$

In this section, we choose the design parameter Λ in the coordinate transformation in Eq. (10) as $\Lambda = k_\Lambda I_2$ with a positive constant $k_\Lambda > 0$. Then, the matrix $T(q)$ defined in Eq. (11) is given by $T(q) = k_\Lambda M(q)^{-1/2}$. Also, we choose the assignable potential energy $U(q)$ in Eq. (12) as $U(q) = 1/2 q^\top \Lambda^\top Q \Lambda q$ with a positive definite constant matrix $Q \in \mathbb{R}^{2\times 2}$. It implies that $\bar{U}(\bar{q})$ in Eq. (14) is given by $\bar{U}(\bar{q}) = 1/2\bar{q}^\top Q\bar{q}$. From the definition (42), we have

$$\begin{aligned} \frac{\partial \lambda_1}{\partial q} &= \Big(0, -(1+\sqrt{2})\mu_2\sin(q_2)\Big), \quad \frac{\partial^2 \lambda_1}{\partial q^2} = \begin{pmatrix} 0 & 0 \\ 0 & -(1+\sqrt{2})\mu_2\cos(q_2) \end{pmatrix}, \\ \frac{\partial \lambda_2}{\partial q} &= \Big(0, -(1-\sqrt{2})\mu_2\sin(q_2)\Big), \quad \frac{\partial^2 \lambda_2}{\partial q^2} = \begin{pmatrix} 0 & 0 \\ 0 & -(1-\sqrt{2})\mu_2\cos(q_2) \end{pmatrix}. \end{aligned}$$

We define the following notations $F_1(a,b)$ and $F_2(a,b)$ for any $a, b \in \mathbb{R}$:

$$\frac{\partial}{\partial q_2}\left(\frac{a}{\sqrt{\lambda_1}} + \frac{b}{\sqrt{\lambda_2}}\right) = \frac{a}{2}\lambda_1^{-\frac{3}{2}}(1+\sqrt{2})\mu_2\sin(q_2) + \frac{b}{2}\lambda_2^{-\frac{3}{2}}(1-\sqrt{2})\mu_2\sin(q_2) =: F_1(a,b), \tag{44}$$

$$\begin{aligned} \frac{\partial^2}{\partial q_2^2}\left(\frac{a}{\sqrt{\lambda_1}} + \frac{b}{\sqrt{\lambda_2}}\right) &= \frac{3a}{4}\lambda_1^{-\frac{5}{2}}(3+2\sqrt{2})\mu_2^2\sin^2(q_2) + \frac{a}{2}\lambda_1^{-\frac{3}{2}}(1+\sqrt{2})\mu_2\cos(q_2) \\ &+ \frac{3b}{4}\lambda_2^{-\frac{5}{2}}(3-2\sqrt{2})\mu_2^2\sin^2(q_2) + \frac{b}{2}\lambda_2^{-\frac{3}{2}}(1-\sqrt{2})\mu_2\cos(q_2) =: F_2(a,b) \end{aligned} \tag{45}$$

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with λ_1, λ_2 defined in (42). Then, by using Eqs. (43) and (45), the last term in the feedback input given by Eq. (12) is reduced to

$$\begin{aligned}
&\Lambda^\top T(q)^{-1} \begin{pmatrix} \sum_{j,k,l=1}^m \frac{1}{2} \frac{\partial^2 [T\Lambda^{-\top}]_{1,j}}{\partial q_l \partial q_k} p_j [h_{11} h_{11}^\top]_{l,k} \\ \vdots \\ \sum_{j,k,l=1}^m \frac{1}{2} \frac{\partial^2 [T\Lambda^{-\top}]_{m,j}}{\partial q_l \partial q_k} p_j [h_{11} h_{11}^\top]_{l,k} \end{pmatrix} \\
&= M(q)^{\frac{1}{2}} \begin{pmatrix} \frac{1}{2} \left(\frac{\partial^2 [M^{-\frac{1}{2}}]_{1,1}}{\partial q_2^2} p_1 [h_{11} h_{11}^\top]_{2,2} + \frac{\partial^2 [M^{-\frac{1}{2}}]_{1,2}}{\partial q_2^2} p_2 [h_{11} h_{11}^\top]_{2,2} \right) \\ \frac{1}{2} \left(\frac{\partial^2 [M^{-\frac{1}{2}}]_{2,1}}{\partial q_2^2} p_1 [h_{11} h_{11}^\top]_{2,2} + \frac{\partial^2 [M^{-\frac{1}{2}}]_{2,2}}{\partial q_2^2} p_2 [h_{11} h_{11}^\top]_{2,2} \right) \end{pmatrix} \\
&= \frac{1}{2} M(q)^{\frac{1}{2}} \begin{pmatrix} F_2\left(\frac{2+\sqrt{2}}{4}, \frac{2-\sqrt{2}}{4}\right) p_1 [h_{11} h_{11}^\top]_{2,2} + F_2\left(\frac{\sqrt{2}}{4}, \frac{-\sqrt{2}}{4}\right) p_2 [h_{11} h_{11}^\top]_{2,2} \\ F_2\left(\frac{\sqrt{2}}{4}, \frac{-\sqrt{2}}{4}\right) p_1 [h_{11} h_{11}^\top]_{2,2} + F_2\left(\frac{2-\sqrt{2}}{4}, \frac{2+\sqrt{2}}{4}\right) p_2 [h_{11} h_{11}^\top]_{2,2} \end{pmatrix}.
\end{aligned}$$

Besides, it follows from Eqs. (43) and (44) that

$$\frac{\partial (T\Lambda^{-\top} p)}{\partial q} = \frac{\partial (M^{-\frac{1}{2}} p)}{\partial q} = \begin{pmatrix} 0 & F_1\left(\frac{(2+\sqrt{2})p_1 + \sqrt{2}p_2}{4}, \frac{(2-\sqrt{2})p_1 - \sqrt{2}p_2}{4}\right) \\ 0 & F_1\left(\frac{\sqrt{2}p_1 + (2-\sqrt{2})p_2}{4}, \frac{-\sqrt{2}p_1 + (2+\sqrt{2})p_2}{4}\right) \end{pmatrix},$$

which appears in Eq. (16) in the transformed system, and in Assumption 4.

In what follows, we design the SISS controller using Theorem 2, and examine its performance under two scenarios: one is against stochastic noise and state-dependent disturbance; and the other is against stochastic noise and step disturbance. The concrete robot parameters used in the simulation are $m_1 = 0.300$, $m_2 = 0.576$ kg, $l_1 = 0.250, l_{c1} = l_1/4, l_2 = 0.480, l_{c2} = l_2/4$ m and $\bar{I}_1 = 1.61 \times 10^{-3}, \bar{I}_2 = 1.11 \times 10^{-2}$ kg m^2. Under those parameters, μ_1, μ_2 and μ_3 defined in Eq. (41) are $\mu_1 = 5.82 \times 10^{-2}$, $\mu_2 = 1.73 \times 10^{-2}$ and $\mu_3 = 1.94 \times 10^{-2}$, respectively, and thus Assumption 5 is sufficiently satisfied. We consider the following noise port in Eq. (2), which vanishes at the origin:

$$h_{11} := \begin{pmatrix} \sin(\frac{q_1}{2})^2 & 0 \\ 0 & \sin(\frac{q_2}{2})^2 \end{pmatrix}, \quad h_{22} := \begin{pmatrix} \dot{q}_1 & 0 \\ 0 & \dot{q}_2 \end{pmatrix} \tag{46}$$

with the standard Wiener processes $w_1(t), w_2(t) \in \mathbb{R}^2$. $h_{11}\,dw_1$ describes the measurement noise, which becomes large as the rotational angle is far from the origin with the cycle 2π. Besides, $h_{22}\,dw_2$ describes uncertainty in viscous friction. From Eq. (42), we have

$$\begin{aligned}
\lambda_{\min}(M(q)^{\frac{1}{2}}) &= \sqrt{\lambda_2(M)} \geq \sqrt{(2-\sqrt{2})\mu_3 + (1-\sqrt{2})\mu_2} =: \underline{K}_{M^{\frac{1}{2}}}, \\
\lambda_{\max}(M(q)^{\frac{1}{2}}) &= \sqrt{\lambda_1(M)} \leq \sqrt{(2+\sqrt{2})\mu_3 + (1+\sqrt{2})\mu_2} =: \overline{K}_{M^{\frac{1}{2}}}.
\end{aligned}$$

Thus, the estimates in Assumption 1 are obtained as $\underline{K}_{M^{\frac{1}{2}}} = 6.48 \times 10^{-2}$ and $\overline{K}_{M^{\frac{1}{2}}} = 0.329$. Next, we numerically calculate the estimates L_{q11}, L_{p11}, δ_{11}, L_{q22}, L_{p22} and δ_{22} in Assumption 2, and L_{q21}, L_{p21} and δ_{21} in Assumption 4, in the range of $-2\pi \leq q_1, q_2 \leq 2\pi$, $-2.0 \leq p_1 \leq 2.0$ and $-1.0 \leq p_2 \leq 1.0$, respectively. The results are $L_{q11} = 0.132$, $L_{p21} = 3.60$, $L_{p22} = 1.00$, and

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$L_{p11} = \delta_{11} = L_{q22} = \delta_{22} = L_{q21} = \delta_{21} = 0$. We set design parameters as $Q = k_Q I_2$, $R_1 = k_{R_1} I_2$, $R_2 = k_{R_2} I_2$ and $R_3 = k_{R_3} I_2$, respectively.

Under the setting, the conditions (28), (29) and (30) for SISS in Theorem 2 are reduced to

$$\begin{aligned} \frac{k_\Lambda k_Q k_{R_1}}{\overline{K}_{M^{\frac{1}{2}}}} &> 2(1 + 8k_Q {k_{R_1}}^2) L_{q11} + 3{\overline{K}_{M^{\frac{1}{2}}}}^2 k_Q {k_{R_1}}^2 \left(L_{p21} + \frac{L_{p22}}{(\underline{K}_{M^{\frac{1}{2}}})^2} \right) \\ k_{R_2} &> 3(\overline{K}_{M^{\frac{1}{2}}})^2 L_{p21} + \frac{1 + 6(\overline{K}_{M^{\frac{1}{2}}})^2 L_{p22}}{(2\underline{K}_{M^{\frac{1}{2}}})^2} \\ k_{R_3} &> 3(\overline{K}_{M^{\frac{1}{2}}})^2 \left(L_{p21} + \frac{L_{p22}}{(\underline{K}_{M^{\frac{1}{2}}})^2} \right). \end{aligned} \tag{47}$$

Note that we can always choose the design parameters k_Λ, k_Q, k_{R_1}, k_{R_2} and k_{R_3} such that the conditions (47) are satisfied. A policy of choosing those parameters is as follows. Since k_Q is the gain of the assigned potential energy, we let k_Q be fairly large to achieve fast-response. However, since the product $k_Q k_{R_1}^2$ appears in the right hand side of the first condition, we let k_{R_1} be small enough. Then, by letting k_Λ be large enough, the first condition holds. The rest of the conditions are easily satisfied by letting k_{R_2} and k_{R_3} be large enough, respectively. We empirically choose the design parameters as $k_Q = 3$, $k_\Lambda = 2$, $k_{R_1} = 5.0 \times 10^{-2}$, $k_{R_2} = 200$ and $k_{R_3} = 80$ so that all of the conditions (47) hold.

4.1. SISS controller design for stochastic noise and state-dependent disturbance

In the first scenario, we consider the following state-dependent bounded disturbance v:

$$v(t) = \begin{cases} 0_{2\times 1} & (0 \le t < 4) \\ \begin{pmatrix} k_{v1} \tanh(\dot{q}_1(t)) \\ k_{v2} \tanh(\dot{q}_2(t)) \end{pmatrix} & (4 \le t \le 7) \\ 0_{2\times 1} & (7 < t) \end{cases} \tag{48}$$

with $k_{v1} = 3$ and $k_{v2} = 1$. Here, we observe the performance of the SISS controller under the noise port (46) and the external disturbance (48).

We set the initial state as $q_1(0) = -\pi, q_2(0) = 0$ rad and $\dot{q}_1(0) = \dot{q}_2(0) = 0$ rad/s. The initial state of the integrator dynamics in (17) is chosen as $\bar{r}(0) = 0_{2\times 1}$. The simulation is executed on $t \in [0, 10]$ s, where during the time interval $4 \le t \le 7$, the external disturbance v is active as in Eq. (48). The simulation results are shown in Figs. 2-6. Figure 2 shows the state-dependent disturbance $v = (v_1, v_2)^\top$ defined in Eq. (48). Figures 3 and 4 respectively exhibit the responses of the joint angles q_1 and q_2, and those of the momentum p_1 and p_2. Figure 5 denotes the responses of the integrator states $\bar{r}_1$ and $\bar{r}_2$. Those figures show that until the external disturbance v is injected at 4s, the proposed controller compensates the stochastic noise effects, and thus the state is approaching to the origin. Even when the external disturbance exists, the state does not diverge and it remains around the origin. After the external disturbance is removed, the state stochastically converges to the origin. Figure 6 shows the time history of the control input $u = (u_1, u_2)^\top$ given by Eq. (36).

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Although the simulation is executed on a finite time interval, those figures imply that the designed controller achieves the control objective.

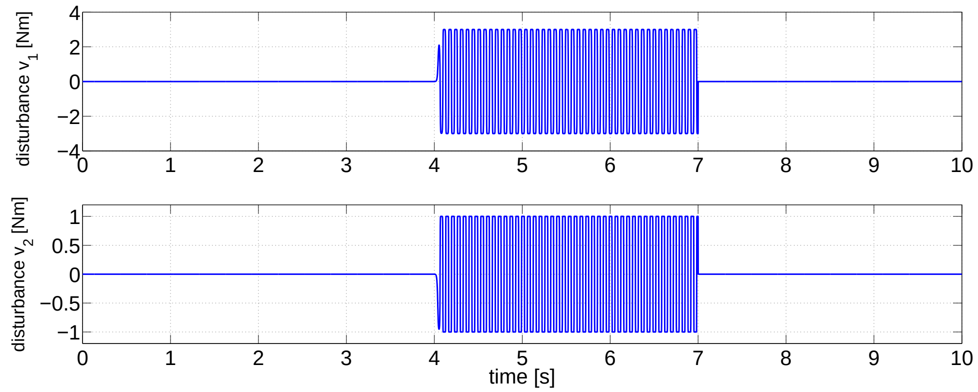

Figure 2. The external disturbances v_1 and v_2

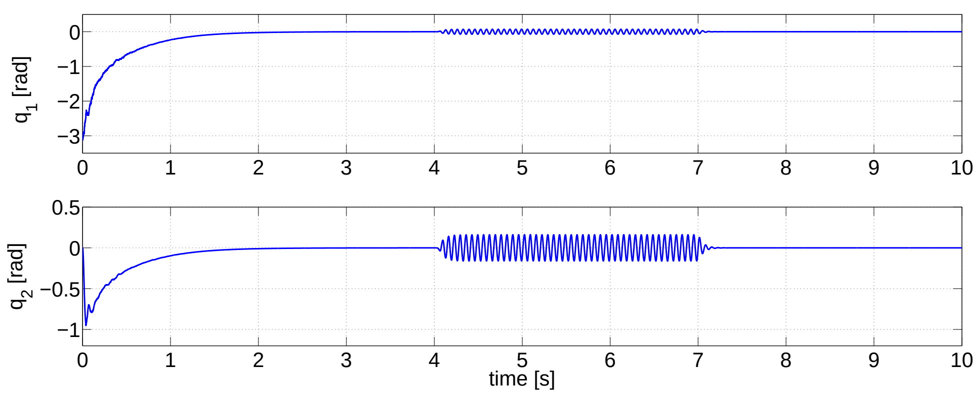

Figure 3. The joint angles q_1 and q_2

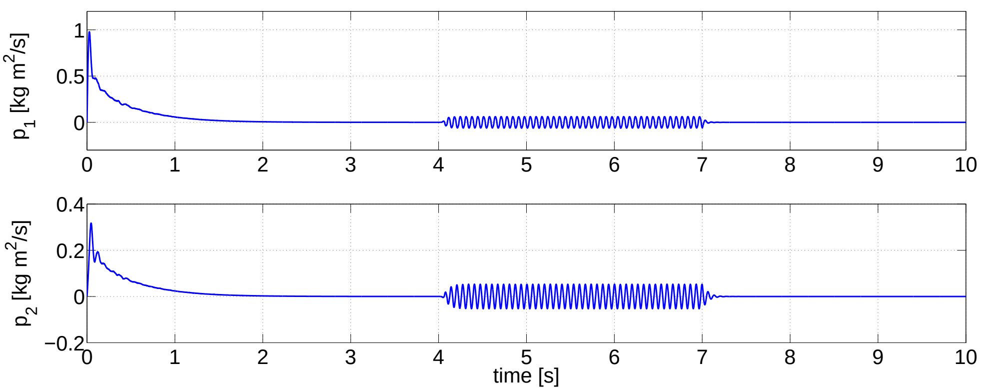

Figure 4. The generalized moments p_1 and p_2

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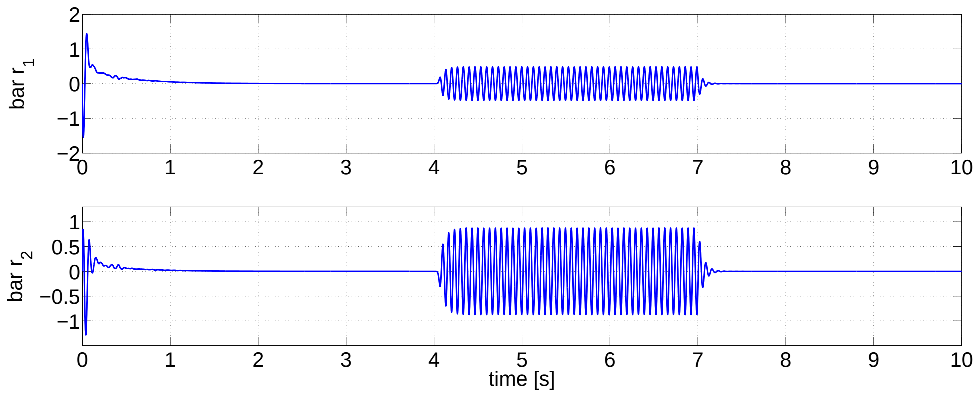

Figure 5. The integrator states $\bar{r}_1$ and $\bar{r}_2$

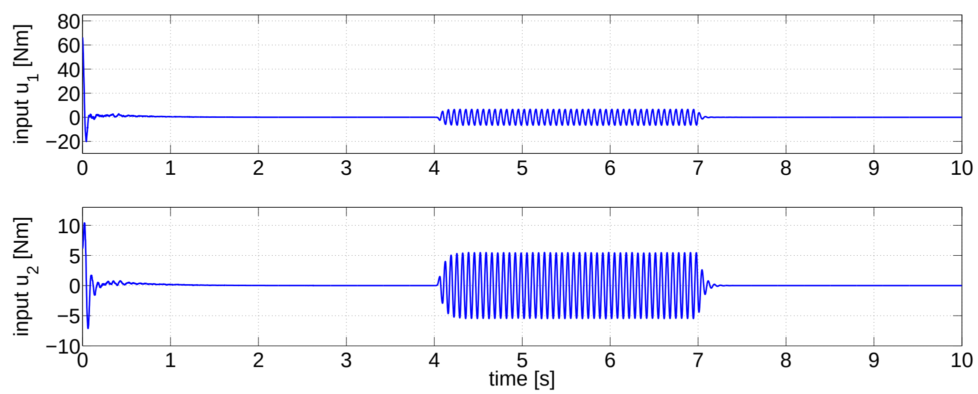

Figure 6. The control inputs u_1 and u_2

4.2. *SISS controller design for stochastic noise and step disturbance*

In the second scenario, we equip the same noise port as in (46), while we consider the following step disturbance v:

$$v(t) = \begin{cases} 0_{2\times 1} & (0 \le t < 4) \\ (k_{v1}, k_{v2})^\top & (4 \le t \le 7) \\ 0_{2\times 1} & (7 < t) \end{cases} \tag{49}$$

with $k_{v1} = 50$ and $k_{v2} = 30$. Now, we use the same SISS controller as in Subsection 4.1, and examine the performance under the noise port (46) and the external disturbance (49).

We use the same initial state as in Subsection 4.1, that is, $q_1(0) = -\pi, q_2(0) = 0$ rad and $\dot{q}_1(0) = \dot{q}_2(0) = 0$ rad/s. The initial state of the integrator dynamics in (17) is also chosen as $\bar{r}(0) = 0_{2\times 1}$. The simulation is executed on $t \in [0, 10]$ s, where during the time interval $4 \le t \le 7$, the external disturbance v is active as in Eq. (49). The simulation results are shown in Figs. 7-11.

Figure 7 shows the step disturbance $v = (v_1, v_2)^\top$ defined in Eq. (49). Figures 8 and 9 respectively exhibit the responses of the joint angles q_1 and q_2, and those of the momentum p_1 and p_2. Figure 10 denotes the responses of the integrator states $\bar{r}_1$ and $\bar{r}_2$. Figure 11 shows the time history of the control input $u = (u_1, u_2)^\top$ given by Eq. (36). As is the case in Subsection 4.1, those figures show that until the external disturbance v is injected at 4s, the proposed controller compensates the stochastic noise effects, and thus the state is approaching to the origin. Even when the external disturbance exists, the state does not diverge and it remains around the origin. After the external disturbance is removed, the state stochastically converges to the origin.

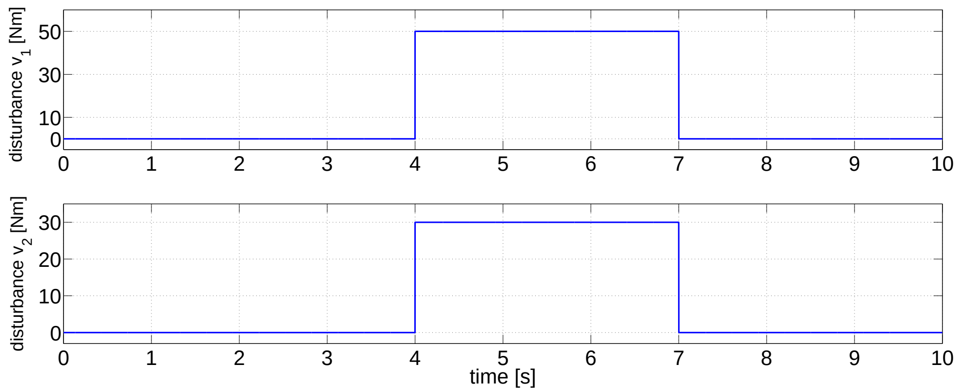

Figure 7. The external disturbances v_1 and v_2

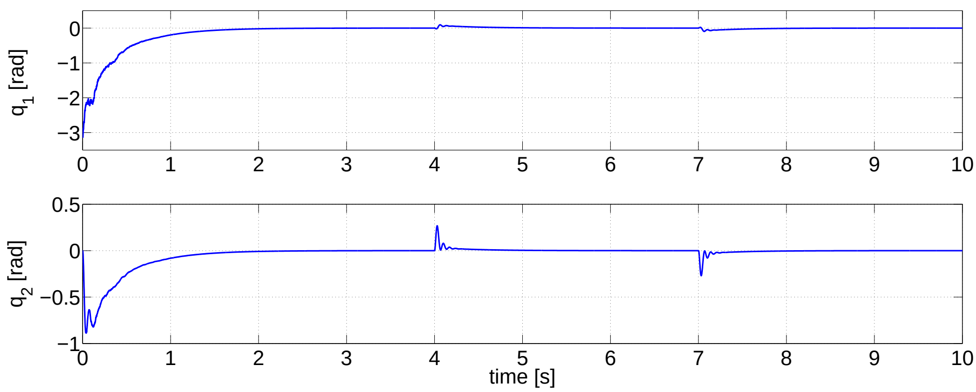

Figure 8. The joint angles q_1 and q_2

In both cases, the designed controller based on Theorem 2 achieves the SISS property against the stochastic noises and deterministic disturbances. Those simulation results demonstrate the validity of the proposed method.

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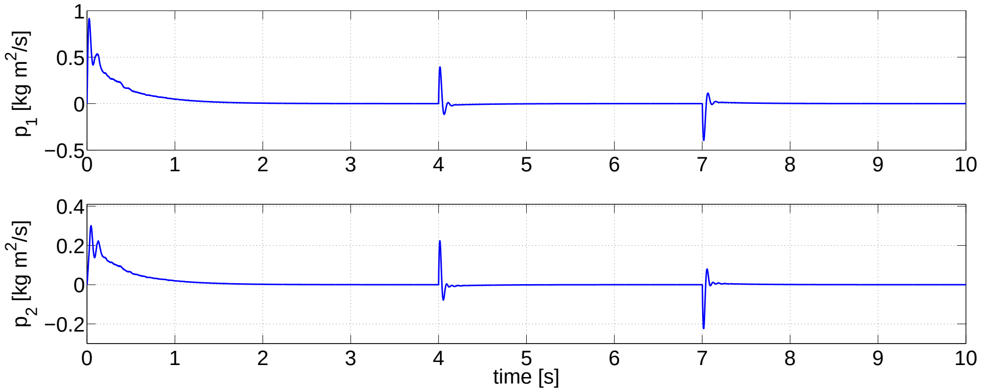

Figure 9. The generalized moments p_1 and p_2

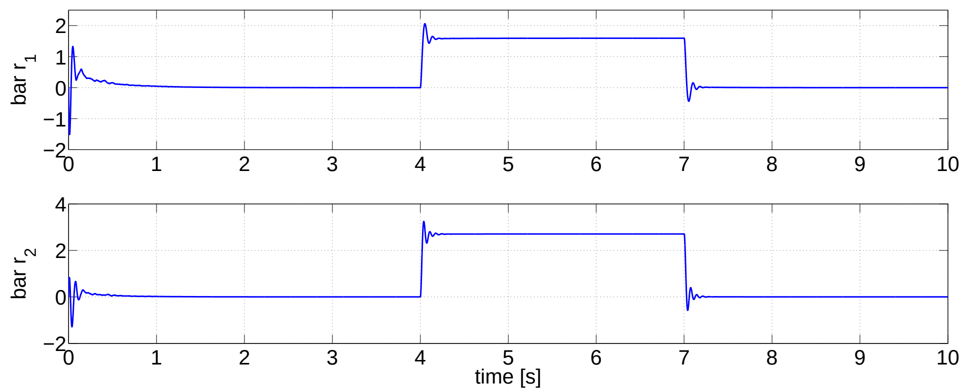

Figure 10. The integrator states $\bar{r}_1$ and $\bar{r}_2$

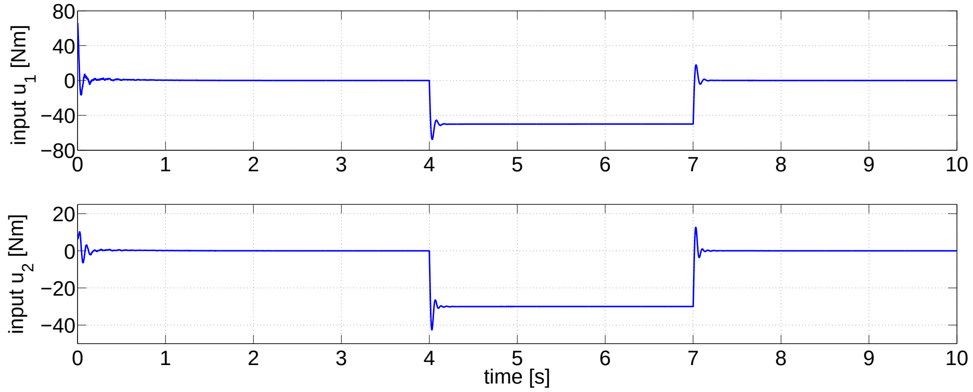

Figure 11. The control inputs u_1 and u_2

5. CONCLUSION

As a practically important class of nonlinear stochastic systems, this paper has considered stochastic port-Hamiltonian systems (SPHSs), and has investigated the stochastic input-to-state stability (SISS) property of a class of SPHSs. We have shown necessary conditions for the closed-loop system to be SISS. Moreover, we have provided a step-by-step construction of both the SISS controller and Lyapunov function so that the proposed necessary conditions hold. In the main results, the stochastic generalized canonical transformation (SGCT) plays a key role. The SGCT technique enables to design both coordinate transformation and feedback controller with preserving the SPHS structure of the closed-loop system. Consequently, the main theorem has guaranteed that the closed-loop system with the proposed controller is SISS against both deterministic disturbance and stochastic noise.

A. PROOF OF PROPOSITION 2

First, by using Theorem 1, we show that a pair of the coordinate transformation (10) and feedback transformation (12) is an SGCT, namely it preserves the SPHS structure of the form (1). Let us start with the left hand side of the condition (9), that is

$$\frac{1}{2}\operatorname{tr}\left\{\frac{\partial^2\Phi_i}{\partial x^2}hh^\top\right\} = \frac{1}{2}\operatorname{tr}\left\{\frac{\partial^2\Phi_i}{\partial q^2}h_{11}h_{11}^\top\right\} + \frac{1}{2}\operatorname{tr}\left\{\frac{\partial^2\Phi_i}{\partial p^2}h_{22}h_{22}^\top\right\}. \tag{50}$$

From Eq. (10), we have

$$\frac{\partial\Phi(x)}{\partial x} = \begin{pmatrix} \Lambda & 0_{m\times m} \\ \frac{\partial(T(q)\Lambda^{-\top}p)}{\partial q} & T(q)\Lambda^{-\top} \end{pmatrix}. \tag{51}$$

Equation (51) and the fact that Λ is a constant matrix lead that

$$\frac{1}{2}\operatorname{tr}\left\{\frac{\partial^2\Phi_i}{\partial x^2}hh^\top\right\} = 0, \quad \text{for } 1 \le i \le m. \tag{52}$$

In the case of $m+1 \le i \le 2m$, we have

$$\left[\frac{\partial^2\Phi_i}{\partial q^2}\right]_{k,l} = \sum_{j=1}^{m}\frac{\partial^2[T\Lambda^{-\top}]_{i-m,j}}{\partial q_l \partial q_k}p_j, \tag{53}$$

$$\frac{\partial^2\Phi_i}{\partial p^2} = 0_{m\times m} \tag{54}$$

and thus, it follows from Eqs. (50), (53) and (54) that

$$\frac{1}{2}\operatorname{tr}\left\{\frac{\partial^2\Phi_i}{\partial x^2}hh^\top\right\} = \sum_{j,k,l=1}^{m}\frac{1}{2}\frac{\partial^2[T\Lambda^{-\top}]_{i-m,j}}{\partial q_l \partial q_k}p_j[h_{11}h_{11}^\top]_{l,k}, \quad \text{for } m+1 \le i \le 2m. \tag{55}$$

Besides, by choosing $P = Q = 0_{m\times m}$, the right hand side (R.H.S) of the condition (9) becomes

$$\text{R.H.S. of (9)} = \begin{pmatrix} 0_{m,1} \\ T(q)\Lambda^{-\top}\left(-\frac{\partial(U(q)-U_0(q))}{\partial q}^{\top} + \beta(q,p)\right) \end{pmatrix}. \tag{56}$$

According to Eqs. (52), (55) and (56), we can determine $\beta(q,p)$ so that the condition (9) is satisfied, as

$$\beta(q,p) = \frac{\partial(U(q)-U_0(q))}{\partial q}^{\top} + \Lambda^{\top}T(q)^{-1}\begin{pmatrix} \sum_{j,k,l=1}^{m} \frac{1}{2}\frac{\partial^2[T\Lambda^{-\top}]_{1,j}}{\partial q_l \partial q_k} p_j [h_{11}h_{11}^{\top}]_{l,k} \\ \vdots \\ \sum_{j,k,l=1}^{m} \frac{1}{2}\frac{\partial^2[T\Lambda^{-\top}]_{m,j}}{\partial q_l \partial q_k} p_j [h_{11}h_{11}^{\top}]_{l,k} \end{pmatrix}. \tag{57}$$

Then, the resultant feedback input $\bar{u}$ given by Eq. (8) coincides with Eq. (12). Therefore, Theorem 1 concludes that the pair of transformations (10) and (12) forms an SGCT.

Next, we show that the transformed system is given by Eq. (13). By utilizing Itô's formula [18, 19], the dynamics of the system in the new coordinate is calculated as

$$\begin{aligned}\begin{pmatrix} \mathrm{d}\bar{q} \\ \mathrm{d}\bar{p} \end{pmatrix} &= \begin{pmatrix} \Lambda & 0 \\ \frac{\partial(T\Lambda^{-\top}p)}{\partial q} & T\Lambda^{-\top} \end{pmatrix}\left\{\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\frac{\partial H}{\partial x}^{\top}\mathrm{d}t + \begin{pmatrix} 0 \\ \bar{u}-\beta \end{pmatrix}\mathrm{d}t + \begin{pmatrix} 0 \\ v \end{pmatrix}\mathrm{d}t + \begin{pmatrix} h_{11}\,\mathrm{d}w_1 \\ h_{22}\,\mathrm{d}w_2 \end{pmatrix}\right\} \\ &\quad + \frac{1}{2}\begin{pmatrix} \mathrm{tr}\left\{\frac{\partial^2\Phi_1}{\partial q^2}h_{11}h_{11}^{\top}\right\} + \mathrm{tr}\left\{\frac{\partial^2\Phi_1}{\partial p^2}h_{22}h_{22}^{\top}\right\} \\ \vdots \\ \mathrm{tr}\left\{\frac{\partial^2\Phi_{2m}}{\partial q^2}h_{11}h_{11}^{\top}\right\} + \mathrm{tr}\left\{\frac{\partial^2\Phi_{2m}}{\partial p^2}h_{22}h_{22}^{\top}\right\} \end{pmatrix}\mathrm{d}t. \end{aligned} \tag{58}$$

By substituting β specified in Eq. (57) for Eq. (58), Eq. (58) is reduced to

$$\begin{aligned}\begin{pmatrix} \mathrm{d}\bar{q} \\ \mathrm{d}\bar{p} \end{pmatrix} &= \begin{pmatrix} 0 & \Lambda \\ -T\Lambda^{-\top} & \frac{\partial(T\Lambda^{-\top}p)}{\partial q} \end{pmatrix}\frac{\partial(H+U-U_0)}{\partial x}^{\top}\mathrm{d}t + \begin{pmatrix} 0 \\ T\Lambda^{-\top}\bar{u} \end{pmatrix}\mathrm{d}t + \begin{pmatrix} 0 \\ T\Lambda^{-\top}v \end{pmatrix}\mathrm{d}t \\ &\quad + \begin{pmatrix} \Lambda h_{11} & 0 \\ \frac{\partial(T\Lambda^{-\top}p)}{\partial q}h_{11} & T\Lambda^{-\top}h_{22} \end{pmatrix}\begin{pmatrix} \mathrm{d}w_1 \\ \mathrm{d}w_2 \end{pmatrix}. \end{aligned} \tag{59}$$

By using the relation

$$\frac{\partial(H+U-U_0)(x)}{\partial x} = \frac{\partial(H+U-U_0)(\Phi^{-1}(\bar{x}))}{\partial \bar{x}}\frac{\partial \Phi}{\partial x},$$

we can observe that the system (59) is the SPHS of the form (13), where the new Hamiltonian is given by

$$\begin{aligned}\bar{H}(\bar{q},\bar{p}) &= (H - U_0 + U)|_{x=\Phi^{-1}(\bar{x})} \\ &= \frac{1}{2}\bar{p}^{\top}T^{-1}\Lambda M^{-1}\Lambda^{\top}T^{-1}\bar{p} + U(\Lambda^{-1}\bar{q}), \end{aligned} \tag{60}$$

which implies Eq. (14) with T defined in Eq. (11).

Therefore, the assertion of Proposition 2 has been proved.

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B. PROOF OF PROPOSITION 3

First, we suppose the feedback controller $\hat{u}$ of the form

$$\hat{u} = -\left(R_1 \frac{\partial^2 \bar{U}}{\partial \bar{q}^2} T + J_2 + R_2 + R_3\right)\bar{p} - (R_2 + R_3)\bar{r} - (T + (R_2 + R_3)R_1)\frac{\partial \bar{U}}{\partial \bar{q}}^\top + \hat{\beta}, \tag{61}$$

where $\hat{\beta}$ is later specified to compensate the noise effect in order to preserve the SPHS structure. The extended system of (13) with the integrator dynamics (17) and the feedback controller (61) is given by

$$\begin{pmatrix} \mathrm{d}\bar{q} \\ \mathrm{d}\bar{p} \\ \mathrm{d}\bar{r} \end{pmatrix} = \begin{pmatrix} T\bar{p} \\ -(2T+(R_2+R_3)R_1)\frac{\partial \bar{U}}{\partial \bar{q}}^\top - \left(R_1 \frac{\partial^2 \bar{U}}{\partial \bar{q}^2} T + R_2 + R_3\right)\bar{p} - (R_2+R_3)\bar{r} \\ (T + R_3 R_1)\frac{\partial \bar{U}}{\partial \bar{q}}^\top + R_3 \bar{p} \end{pmatrix} \mathrm{d}t + \begin{pmatrix} 0 \\ \hat{\beta} \\ 0 \end{pmatrix} \mathrm{d}t$$
$$+ \begin{pmatrix} 0 \\ \bar{k}_2 \\ 0 \end{pmatrix} v\,\mathrm{d}t + \begin{pmatrix} \bar{h}_{11} & 0 \\ \bar{h}_{21} & \bar{h}_{22} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}w_1 \\ \mathrm{d}w_2 \end{pmatrix}. \tag{62}$$

Here, we define $\bar{h}$ as

$$\bar{h}(\bar{q}, \bar{p}) := \begin{pmatrix} \bar{h}_{11}(\bar{q}, \bar{p}) & 0_{m \times m_{w2}} \\ \bar{h}_{21}(\bar{q}, \bar{p}) & \bar{h}_{22}(\bar{q}, \bar{p}) \\ 0_{m \times m_{w1}} & 0_{m \times m_{w2}} \end{pmatrix}. \tag{63}$$

Then, by Itô's formula [18, 19], the system (62) is described in the new coordinate z in (19) as

$$\begin{pmatrix} \mathrm{d}z_1 \\ \mathrm{d}z_2 \\ \mathrm{d}z_3 \end{pmatrix} = \frac{\partial \Psi(\bar{q}, \bar{p}, \bar{r})}{\partial (\bar{q}, \bar{p}, \bar{r})} \begin{pmatrix} \mathrm{d}\bar{q} \\ \mathrm{d}\bar{p} \\ \mathrm{d}\bar{r} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \mathrm{tr}\left\{ \frac{\partial^2 \Psi_1}{\partial (\bar{q},\bar{p},\bar{r})^2} \bar{h}\bar{h}^\top \right\} \\ \vdots \\ \mathrm{tr}\left\{ \frac{\partial^2 \Psi_{3m}}{\partial (\bar{q},\bar{p},\bar{r})^2} \bar{h}\bar{h}^\top \right\} \end{pmatrix} \mathrm{d}t. \tag{64}$$

Since from Eq. (19), we have

$$\frac{\partial \Psi(\bar{q}, \bar{p}, \bar{r})}{\partial (\bar{q}, \bar{p}, \bar{r})} = \begin{pmatrix} I_m & 0_{m \times m} & 0_{m \times m} \\ R_1 \frac{\partial^2 \bar{U}(\bar{q})}{\partial \bar{q}^2} & I_m & I_m \\ 0_{m \times m} & 0_{m \times m} & I_m \end{pmatrix}, \tag{65}$$

$$\frac{1}{2}\,\mathrm{tr}\left\{ \frac{\partial^2 \Psi_i}{\partial (\bar{q}, \bar{p}, \bar{r})^2} \bar{h}\bar{h}^\top \right\} = 0, \quad \text{for } 1 \le i \le m, \quad 2m \le i \le 3m. \tag{66}$$

In the case of $m+1 \leq i \leq 2m$, we have

$$\left[\frac{\partial^2 \Psi_i}{\partial \bar{q}^2}\right]_{k,l} = \sum_{j=1}^{m} [R_1]_{i-m,j} \frac{\partial^3 \bar{U}(\bar{q})}{\partial \bar{q}_l \partial \bar{q}_k \partial \bar{q}_j}, \tag{67}$$

$$\frac{\partial^2 \Psi_i}{\partial \bar{p}^2} = \frac{\partial^2 \Psi_i}{\partial \bar{r}^2} = 0_{m\times m}. \tag{68}$$

It follows from Eqs. (63), (65), (67) and (68) that

$$\frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial^2 \Psi_i}{\partial(\bar{q},\bar{p},\bar{r})^2}\bar{h}\bar{h}^\top\right\} = \sum_{j,k,l=1}^{m} \frac{1}{2}[R_1]_{i-m,j}\frac{\partial^3 \bar{U}(\bar{q})}{\partial \bar{q}_l \partial \bar{q}_k \partial \bar{q}_j}[\bar{h}_{11}\bar{h}_{11}^\top]_{l,k}, \qquad \text{for } m+1 \leq i \leq 2m. \tag{69}$$

Equations (66) and (69) imply that

$$\frac{1}{2}\begin{pmatrix} \mathrm{tr}\left\{\frac{\partial^2 \Psi_1}{\partial(\bar{q},\bar{p},\bar{r})^2}\bar{h}\bar{h}^\top\right\} \\ \vdots \\ \mathrm{tr}\left\{\frac{\partial^2 \Psi_{3m}}{\partial(\bar{q},\bar{p},\bar{r})^2}\bar{h}\bar{h}^\top\right\} \end{pmatrix} = \begin{pmatrix} 0_{m\times m} \\ I_m \\ 0_{m\times m} \end{pmatrix} \begin{pmatrix} \sum_{j,k,l=1}^{m} \frac{1}{2}[R_1]_{1,j}\frac{\partial^3 \bar{U}(\bar{q})}{\partial \bar{q}_l \partial \bar{q}_k \partial \bar{q}_j}[\bar{h}_{11}\bar{h}_{11}^\top]_{l,k} \\ \vdots \\ \sum_{j,k,l=1}^{m} \frac{1}{2}[R_1]_{m,j}\frac{\partial^3 \bar{U}(\bar{q})}{\partial \bar{q}_l \partial \bar{q}_k \partial \bar{q}_j}[\bar{h}_{11}\bar{h}_{11}^\top]_{l,k} \end{pmatrix}.$$

Thus, we can determine the input transformation $\hat{\beta}$ in (62) as

$$\hat{\beta} = -\begin{pmatrix} \sum_{j,k,l=1}^{m} \frac{1}{2}[R_1]_{1,j}\frac{\partial^3 \bar{U}(\bar{q})}{\partial \bar{q}_l \partial \bar{q}_k \partial \bar{q}_j}[\bar{h}_{11}\bar{h}_{11}^\top]_{l,k} \\ \vdots \\ \sum_{j,k,l=1}^{m} \frac{1}{2}[R_1]_{m,j}\frac{\partial^3 \bar{U}(\bar{q})}{\partial \bar{q}_l \partial \bar{q}_k \partial \bar{q}_j}[\bar{h}_{11}\bar{h}_{11}^\top]_{l,k} \end{pmatrix}. \tag{70}$$

The resultant controller $\hat{u}$ in (61) with $\hat{\beta}$ in (70) coincides with the controller (18). It is easily verified that a pair of the coordinate and feedback transformations (18) and (19) satisfies the condition in Theorem 1. By substituting Eq. (70) with the transformed system (64), we obtain the system (20).

This proves the assertion of Proposition 3.

C. PROOF OF LEMMA 1

From Eqs. (21) and (22), we have

$$\frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial^2 H_z}{\partial z^2}h_z h_z^\top\right\} = \frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial^2 \bar{U}(z_1)}{\partial z_1^2}h_{z11}h_{z11}^\top\right\} + \frac{1}{2}\,\mathrm{tr}\left\{h_{z21}h_{z21}^\top\right\} + \frac{1}{2}\,\mathrm{tr}\left\{h_{z22}h_{z22}^\top\right\}. \tag{71}$$

Under Assumptions 2 and 3 with Eqs (16) and (22), the following estimation is obtained:

$$\begin{aligned}\frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial^2 \bar{U}(z_1)}{\partial z_1^2}h_{z11}h_{z11}^\top\right\} &= \frac{1}{2}\left\|\left(\frac{\partial^2 \bar{U}}{\partial \bar{q}^2}\right)^{\frac{1}{2}}\Lambda h_{11}\right\|_F^2 \leq \frac{1}{2}\left\|\left(\frac{\partial^2 \bar{U}}{\partial \bar{q}^2}\right)^{\frac{1}{2}}\right\|_F^2 \|\Lambda\|_F^2\|h_{11}\|_F^2 \\ &\leq \frac{1}{2}\overline{K}_{U2}\|\Lambda\|_F^2\left(L_{q11}\|q\|^2 + L_{p11}\|p\|^2 + \delta_{11}^2\right),\end{aligned} \tag{72}$$

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where the sub-multiplicativity of the Frobenius norm is utilized. Similarly, Assumptions 2, 3 and 4 imply the following estimation:

$$\begin{aligned}
\frac{1}{2}\operatorname{tr}\left\{h_{z21}h_{z21}^\top\right\} &= \frac{1}{2}\left\|R_1\frac{\partial^2\bar{U}}{\partial z_1^2}\Lambda h_{11} + \frac{\partial(T\Lambda^{-\top}p)}{\partial q}h_{11}\right\|_F^2 \\
&\le \left\|R_1\frac{\partial^2\bar{U}}{\partial z_1^2}\Lambda h_{11}\right\|_F^2 + \left\|\frac{\partial(T\Lambda^{-\top}p)}{\partial q}h_{11}\right\|_F^2 \\
&\le \|R_1\|_F^2\left\|\frac{\partial^2\bar{U}}{\partial z_1^2}\right\|_F^2\|\Lambda\|_F^2\|h_{11}\|_F^2 + \left\|\frac{\partial(T\Lambda^{-\top}p)}{\partial q}h_{11}\right\|_F^2 \\
&\le (\overline{K}_{U2}{}^2\|R_1\|_F^2\|\Lambda\|_F^2 L_{q11} + L_{q21})\|q\|^2 + (\overline{K}_{U2}{}^2\|R_1\|_F^2\|\Lambda\|_F^2 L_{p11} + L_{p21})\|p\|^2 \\
&\quad + \overline{K}_{U2}{}^2\|R_1\|_F^2\|\Lambda\|_F^2\delta_{11}^2 + \delta_{21}^2. \qquad (73)
\end{aligned}$$

Also, Assumptions 1 and 2 imply the following estimation:

$$\begin{aligned}
\frac{1}{2}\operatorname{tr}\left\{h_{z22}h_{z22}^\top\right\} &\le \frac{1}{2}\|T\Lambda^{-\top}\|_F^2\|h_{22}\|_F^2 = \frac{1}{2}\|M^{-\frac{1}{2}}\|_F^2\|\|h_{22}\|_F^2 \\
&\le \frac{m}{2(\underline{K}_{M^{\frac{1}{2}}})^2}\left(L_{q22}\|q\|^2 + L_{p22}\|p\|^2 + \delta_{22}^2\right), \qquad (74)
\end{aligned}$$

where the definition of T in Eq. (11) is utilized in the second equality, and the norm inequality $\|A\|_F \le \sqrt{\min\{m,n\}}\|A\|$ for a matrix $A \in \mathbb{R}^{m\times n}$ is utilized in the third inequality. It comes from Eqs. (71), (72), (73) and (74) that

$$\begin{aligned}
\frac{1}{2}\operatorname{tr}\left\{\frac{\partial^2 H_z}{\partial z^2}h_z h_z^\top\right\} \le &\left(\frac{\overline{K}_{U2}\|\Lambda\|_F^2 L_{q11}}{2}(1+2\overline{K}_{U2}\|R_1\|_F^2) + L_{q21} + \frac{mL_{q22}}{2(\underline{K}_{M^{\frac{1}{2}}})^2}\right)\|q\|^2 \\
&+ \left(\frac{\overline{K}_{U2}\|\Lambda\|_F^2 L_{p11}}{2}(1+2\overline{K}_{U2}\|R_1\|_F^2) + mL_{p21} + \frac{mL_{p22}}{2(\underline{K}_{M^{\frac{1}{2}}})^2}\right)\|p\|^2 \\
&+ \frac{\overline{K}_{U2}\|\Lambda\|_F^2}{2}(1+2\overline{K}_{U2}\|R_1\|_F^2)\delta_{11}^2 + \delta_{21}^2 + \frac{m}{2(\underline{K}_{M^{\frac{1}{2}}})^2}\delta_{22}^2. \qquad (75)
\end{aligned}$$

Besides, from the coordinate transformations in (10), (19), we have

$$\|q\|^2 = \|\Lambda^{-1}z_1\|^2 \le \frac{1}{\lambda_{\min}(\Lambda)^2}\|z_1\|^2, \qquad (76)$$

and

$$\begin{aligned}
\|p\|^2 &= \left\|\Lambda^\top T^{-1}\left(-R_1\frac{\partial\bar{U}}{\partial z_1}^\top + z_2 - z_3\right)\right\|^2 \\
&\le \left\|\Lambda^\top T^{-1}\right\|^2\left(3\left\|R_1\frac{\partial\bar{U}}{\partial z_1}^\top\right\|^2 + 3\|z_2\|^2 + 3\|z_3\|^2\right) \\
&\le 3(\overline{K}_{M^{\frac{1}{2}}})^2\left(\lambda_{\max}(R_1)^2\overline{K}_{U1}{}^2\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2\right), \qquad (77)
\end{aligned}$$

where the following relation is used in the last inequality:

$$\left\|\Lambda^\top T^{-1}\right\|^2 = \lambda_{\max}(\Lambda^\top T^{-1}T^{-1}\Lambda) = \lambda_{\max}(M) = \lambda_{\max}(M^{\frac{1}{2}}M^{\frac{1}{2}}) = \|M^{\frac{1}{2}}\|^2.$$

Consequently, by substituting Eqs. (76) and (77) for the estimation (75), we have

$$\begin{aligned}
&\frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial^2 H_z}{\partial z^2}h_z h_z^\top\right\} \\
&\le \Bigg\{\frac{\overline{K}_{U2}\|\Lambda\|_F^2}{2}(1+2\overline{K}_{U2}\|R_1\|_F^2)\left(\frac{L_{q11}}{\lambda_{\min}(\Lambda)^2}+3(\overline{K}_{M^{\frac{1}{2}}})^2\lambda_{\max}(R_1)^2\overline{K}_{U1}{}^2L_{p11}\right) \\
&\quad + \frac{L_{q21}}{\lambda_{\min}(\Lambda)^2}+3(\overline{K}_{M^{\frac{1}{2}}})^2\lambda_{\max}(R_1)^2\overline{K}_{U1}{}^2L_{p21} \\
&\quad + \frac{m}{2(\underline{K}_{M^{\frac{1}{2}}})^2}\left(\frac{L_{q22}}{\lambda_{\min}(\Lambda)^2}+3(\overline{K}_{M^{\frac{1}{2}}})^2\lambda_{\max}(R_1)^2\overline{K}_{U1}{}^2L_{p22}\right)\Bigg\}\|z_1\|^2 \\
&\quad + 3(\overline{K}_{M^{\frac{1}{2}}})^2\left(\frac{\overline{K}_{U2}\|\Lambda\|_F^2}{2}(1+2\overline{K}_{U2}\|R_1\|_F^2)L_{p11}+L_{p21}+\frac{mL_{p22}}{2(\underline{K}_{M^{\frac{1}{2}}})^2}\right)(\|z_2\|^2+\|z_3\|^2) \\
&\quad + \frac{\overline{K}_{U2}\|\Lambda\|_F^2}{2}(1+2\overline{K}_{U2}\|R_1\|_F^2)\delta_{11}^2+\delta_{21}^2+\frac{m\delta_{22}^2}{2(\underline{K}_{M^{\frac{1}{2}}})^2}.
\end{aligned} \tag{78}$$

The resultant estimation (78) proves the assertion of Lemma 1 with L_{z1}, L_{z2}, L_{z3} and δ_z respectively given by Eqs. (25) - (27).

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