Repetitive control of Hamiltonian systems based on variational symmetry

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Abstract

This paper is concerned with repetitive control of Hamiltonian systems, which is based on iterative learning control utilizing the variational symmetry of those systems. Variational symmetry allows us to obtain an algorithm to solve a certain class of optimal control problems in a repetitive control framework. Therefore, the proposed method can deal with not only trajectory tracking control problems but also optimal trajectory generation problems, never before considered in a repetitive control framework. A convergence analysis of this algorithm is also discussed. Furthermore, some numerical simulations demonstrate the effectiveness of the proposed method.

Keywords: nonlinear control, Hamiltonian systems, iterative learning control, repetitive control

1. Introduction

The proportional-derivative (PD)-type iterative learning control (ILC) method proposed in [1] is an algorithm to generate a feedforward input achieving a trajectory tracking control (on a finite time interval) without using precise information of the plant system [1]. Since this algorithm does

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[1] Other types of ILC, e.g., plant inversion methods, H_∞ methods and Q-ILC in [2], generally require model information. Our main interest here is model-free learning methods.

not require a precise model of the plant, it is robust against modeling errors. Many researchers have worked on this topic; see, e.g., [3, 4]. So far, however, the main objective of the method is basically trajectory tracking control problems. Recently, we have proposed a novel ILC method based on the variational symmetry of Hamiltonian systems [5, 6, 7]. In these results, it is proved that the variational systems of Hamiltonian systems are symmetric and this property can be utilized for executing the iterative algorithm for optimal control problems without using precise information of the plant. This method can generate an optimal trajectory without any desired trajectory or finite-dimensional approximation, as a solution of an infinite-dimensional optimal control problem, or, more concretely, an optimal control problem on L_2 signal space.

On the other hand, repetitive control (RC) is also a useful tool which has a close relationship to ILC; see, e.g., [8, 9]. RC is also a kind of a learning method for a trajectory tracking control problem with time-periodic reference trajectories.The main differences between RC and ILC are those between the reference trajectories and the initializations: RC deals with a time-periodic reference trajectory (with the infinite length) and ILC does this on a finite time interval, and it requires the initialization of the state at the start of each period. However, the optimal trajectory generation problem has never been considered in a repetitive control framework.

This paper focuses on repetitive control of Hamiltonian systems and proposes a new RC framework based on variational symmetry. Although this approach has a defect that the reference (or desired) trajectories have to be time symmetric with respect to the period, it also has many advantages. Not only can it track a reference trajectory, but also it can repetitively generate an optimal trajectory as a solution to an optimal control problem without any reference trajectory. This method is applicable to a class of nonlinear systems which includes fully actuated mechanical systems and, furthermore, it does not require the plant model. We also provide a convergence analysis which guarantees the convergence of the output trajectory on the reference or optimal one under certain conditions. Furthermore, numerical simulations of a robot manipulator demonstrate the effectiveness of the proposed method.

2. Iterative learning optimal control of Hamiltonian systems based on variational symmetry

This section briefly refers to some preliminary background.

2.1. Variational symmetry

The plant system considered here is a Hamiltonian system with dissipation Σ with a controlled Hamiltonian $H(x,u)$ as $(x^1, y) = \Sigma(x^0, u)$:

$$\left\{ \begin{array}{lcll} \dot{x} & = & (J-R)\,\dfrac{\partial H(x,u)}{\partial x}^{\mathrm{T}}, & x(t^0) = x^0 \\ y & = & -\dfrac{\partial H(x,u)}{\partial u}^{\mathrm{T}}, & x^1 = x(t^1) \end{array} \right. . \tag{1}$$

Here $x(t) \in \mathbb{R}^n$ and $u(t), y(t) \in \mathbb{R}^m$ describe the state, the input and the output, respectively. The structure matrix $J \in \mathbb{R}^{n\times n}$ and the dissipation matrix $R \in \mathbb{R}^{n\times n}$ are skew symmetric and symmetric positive semi-definite, respectively. The matrix R represents dissipative elements such as friction of mechanical systems and resistance of electric circuits. We suppose that the Hamiltonian $H(x,u)$ is a sufficiently differentiable function. For this system, the following theorem holds.

Theorem 1. *[5] Consider the Hamiltonian system Σ in (1). Suppose that J and R are constant and that there exists a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ satisfying*

$$J \;=\; -TJ\,T^{-1}, \qquad R = TR\,T^{-1} \tag{2}$$

$$\frac{\partial^2 H(x,u)}{\partial (x,u)^2} \;=\; \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix} \frac{\partial^2 H(x,u)}{\partial (x,u)^2} \begin{pmatrix} T^{-1} & 0 \\ 0 & I \end{pmatrix}. \tag{3}$$

Then the Fréchet derivative of Σ is described by another linear Hamiltonian system $(x_v^1, y_v) = \mathrm{d}\Sigma((x^0,u),(x_v^0,u_v))$:

$$\left\{ \begin{array}{lcll} \dot{x} & = & (J-R)\dfrac{\partial H(x,u)}{\partial x}^{\mathrm{T}}, & x(t^0) = x^0 \\ \dot{x}_v & = & (J-R)\dfrac{\partial H_v(x,u,x_v,u_v)}{\partial x_v}^{\mathrm{T}}, & x_v(t^0) = x_v^0 \\ y_v & = & -\dfrac{\partial H_v(x,u,x_v,u_v)}{\partial u_v}^{\mathrm{T}}, & x_v^1 = x_v(t^1) \end{array} \right.$$

with a controlled Hamiltonian $H_v(x,u,x_v,u_v)$ as

$$H_v(x,u,x_v,u_v) = \frac{1}{2} \begin{pmatrix} x_v \\ u_v \end{pmatrix}^{\mathrm{T}} \frac{\partial^2 H(x,u)}{\partial (x,u)^2} \begin{pmatrix} x_v \\ u_v \end{pmatrix}.$$

Furthermore, the adjoint of the variational system with zero initial state $u_a \mapsto y_a = (\mathrm{d}\Sigma^{x^0}(u))^*(u_a)$ *is given by*

$$\begin{cases} \dot{x} &= (J-R)\dfrac{\partial H(x,u)}{\partial x}^{\mathrm{T}} \\ \dot{\bar{x}}_v &= -(J-R)\dfrac{\partial H_v(x,u,\bar{x}_v,u_a)}{\partial \bar{x}_v}^{\mathrm{T}} \\ y_a &= -\dfrac{\partial H_v(x,u,\bar{x}_v,u_a)}{\partial u_a}^{\mathrm{T}} \end{cases} \tag{4}$$

with the initial state $x(t^0) = x^0$ *and the terminal state* $\bar{x}_v(t^1) = 0$*. Suppose moreover that* $J - R$ *is nonsingular. Then the adjoint* $(x_a^1, u_a) \mapsto (x_a^0, y_a) = (\mathrm{d}\Sigma(x^0, u))^*(x_a^1, u_a)$ *is given by the same state-space realization (4) with the initial state* $x(t^0) = x^0$*, the terminal state* $\bar{x}_v(t^1) = -(J-R)T\ x_a^1$*, and* $x_a^0 = -T^{-1}(J-R)^{-1}\bar{x}_v(t^0)$*.*

This theorem reveals that the variational system and its adjoint of a Hamiltonian system in the form of (1) have almost the same state-space realizations. This means that the input-output mapping of the adjoint can be produced by the input-output data of the original Hamiltonian system as

$$\mathcal{R} \circ (\mathrm{d}\Sigma^{x^0}(u))^* \circ \mathcal{R}(v) = \mathrm{d}\Sigma^{\bar{x}^0}(\bar{u})(v) \approx \frac{1}{\epsilon}\left(\Sigma^{\bar{x}^0}(\bar{u}+\epsilon v) - \Sigma^{\bar{x}^0}(\bar{u})\right) \tag{5}$$

provided that appropriate boundary conditions and sufficiently small positive ϵ are selected, where $\mathcal{R}$ is the time reversal operator defined by

$$\mathcal{R}(u)(t - t^0) = u(t^1 - t), \quad \forall t \in [t^0, t^1]. \tag{6}$$

This property is utilized for solving optimal control problems in which the adjoint operator plays an important role.

Remark 1. It is noted that if the system is a *gradient system* [10] which is a nonlinear generalization of a linear symmetric system, that is, $J = 0$, then the assumption (3) in Theorem 1 is automatically satisfied with $T = I$. On the other hand, if the system is conservative, that is, $R = 0$ then it is self-adjoint in the usual sense [5].

2.2. Optimal control via iterative learning

Let us consider the system $\Sigma : X \times U \to X \times Y$ in (1) and a cost function $\Gamma : X^2 \times U \times Y \to \mathbb{R}$ with Hilbert spaces X, U and Y. Typically, $X = \mathbb{R}^n$, $U = L_2^m[t^0, t^1]$, and $Y = L_2^m[t^0, t^1]$. The objective is to find the optimal input $(x^0_\star, u_\star)$ minimizing the cost function $\Gamma(x^0, u, x^1, y)$. In general, however, it is difficult to obtain a global minimum since the cost function Γ is not convex. Hence we try to obtain a local minimum here. We can calculate

$$\begin{aligned}
&\mathrm{d}\left(\Gamma((x^0, u), \Sigma(x^0, u))\right)(\mathrm{d}x^0, \mathrm{d}u)\\
&= \mathrm{d}\Gamma((x^0, u), \Sigma(x^0, u))\left((\mathrm{d}x^0, \mathrm{d}u), \mathrm{d}\Sigma(x^0, u)(\mathrm{d}x^0, \mathrm{d}u)\right)\\
&= \langle \Gamma'((x^0, u), \Sigma(x^0, u)), \begin{pmatrix} \mathrm{id}_{X\times U} \\ \mathrm{d}\Sigma(x^0, u)\end{pmatrix}(\mathrm{d}x^0, \mathrm{d}u)\rangle_{X^2\times U\times Y}\\
&= \langle \left(\mathrm{id}_{X\times U}, (\mathrm{d}\Sigma(x^0, u))^*\right)\Gamma'(x^0, u, x^1, y), (\mathrm{d}x^0, \mathrm{d}u)\rangle_{X\times U},
\end{aligned}$$

where $\mathrm{id}_{X\times U}$ represents the identity map. The well-known Riesz's representation theorem guarantees that there exists a function $\Gamma'(x^0, u, x^1, y)$ as above. Therefore, if the adjoint $(\mathrm{d}\Sigma(x^0, u))^*$ is available, we can reduce the cost function Γ at least to a local minimum by an iteration law with a $K_{(i)} > 0$

$$u_{(i+1)} = u_{(i)} - K_{(i)}\pi_U \circ \left(\mathrm{id}_{X\times U}, (\mathrm{d}\Sigma(x^0_{(i)}, u_{(i)}))^*\right)\Gamma'(x^0_{(i)}, u_{(i)}, x^1_{(i)}, y_{(i)}), \quad (7)$$

where $\pi_{(\cdot)}$ denotes the projection operator onto $(\cdot)$.

The results in the previous subsection enable one to execute this procedure without using the parameters of the plant Σ by the relation (5), provided that Σ is a Hamiltonian system and that the boundary conditions are selected appropriately. In [5, 6, 7], this framework is effectively utilized for ILC (of trajectory tracking) for a 'round trip'-type trajectory. A more precise discussion for optimal control will be presented in the following sections.

3. Main results

In this section, we propose a new algorithm solving a class of repetitive control problems based on the variational symmetry of Hamiltonian systems.

3.1. Repetitive control

A typical repetitive control problem is to achieve trajectory tracking control for a periodic reference trajectory by learning (experiments) without using precise information of the plant [8]. A conventional ILC problem is also to achieve trajectory tracking control by learning but the reference trajectory is defined on a finite time interval, that is, tracking is achieved by several experiments with a finite time interval (with the same initial states) [1].

The approach taken here is to provide a repetitive control framework using the ILC based on variational symmetry. Variational symmetry allows us to obtain an algorithm to solve a certain class of optimal control problems in a repetitive control framework. Here, we adopt the following strategy: suppose that the plant Hamiltonian system is controlled by a feedback designed by generalized canonical transformations [11] so that the closed-loop system is again described by a Hamiltonian system (1). See Section 4 for a concrete example of constructing such a control system and refer to [5] for the details. We make the following assumption on a desired reference trajectory or a desired optimal trajectory.

Assumption 1. The desired reference trajectory or the desired optimal trajectory is L-periodic and time symmetric[2], and it contains a stationary point at the origin.

Then apply the ILC for the first period L from the stationary point and then wait for the state to converge on the stationary point. When the state approaches sufficiently close to the stationary point, the next ILC procedure with time period L starts. Continue in the same manner. A more detailed procedure is explained below. (See Table 1 and Fig. 1 as well.)

Step 0 **:** Suppose that the initial state $x(0)$ is a stationary state. Set $t^0_{(1)} := 0$ and $i = 1$ and then go to Step $1(a)$.

Step i **(a) :** Set $t^L_{(i)} := t^0_{(i)} + L$ and apply ILC input $u_{(i)}$ derived in Section 2 for the time $t \in [t^0_{(i)}, t^L_{(i)}]$.

Then, set $u(t) \equiv 0$ for $t \geq t^L_{(i)}$ and find the smallest $\tau \geq t^L_{(i)}$ satisfying

$$\|x(\tau)\| < b \tag{8}$$

[2] Since ILC based on variational symmetry needs experiments with time-reversal trajectories with respect to the trajectory to be learned, the reference or desired trajectory has to be time symmetric in the repetitive control framework.

with a prescribed constant $b > 0$; $u(t) \equiv 0$ is applied for $t \in [t^L_{(i)}, \tau]$. Define $t^0_{(i+1)} := \tau$ and

$$\Delta t_{(i)} := \tau - t^L_{(i)}. \tag{9}$$

If $i = 1$ or the cost function decreases, then go to Step $i + 1(a)$.

Otherwise, that is, if

$$\begin{aligned} &\mathrm{d}\left(\Gamma((x^0, u), \Sigma(x^0, u))\right) \qquad (10) \\ &\quad \approx \Gamma((x^0, u), \Sigma(x^0, u))\Big|_{\substack{x^0 = x^0_{(i)} \\ u = u_{(i)}}} - \Gamma((x^0, u), \Sigma(x^0, u))\Big|_{\substack{x^0 = x^0_{(i-1)} \\ u = u_{(i-1)}}} \geq 0, \end{aligned}$$

set $i_{stp} = i$, and stop updating the control input. Then go to Step $i + 1(b)$.

Step i **(b) :** Set $t^L_{(i)} := t^0_{(i)} + L$ and apply the last updated control input $u_{(i_{stp})}$ for the time $t \in [t^0_{(i)}, t^L_{(i)}]$.

Then, set $u(t) \equiv 0$ for $t \in [t^L_{(i)}, \tau]$. Define $\Delta t_{(i)} := \tau - t^L_{(i)}$ and $t^0_{(i+1)} := \tau$. Then go to Step $i + 1(b)$.

Since the ILC mentioned in Section 2 is based on the steepest descent method, Eq. (10) implies that a local minimum can be obtained. By this algorithm,

Table 1: Parameters and variables

Notation	Meaning
L	period of the (reference / optimal) trajectory
$t^0_{(i)}$	starting time of the i-th period
$t^L_{(i)}$	terminal time of the i-th period ($t^L_{(i)} = t^0_{(i)} + L$)
$y_{(i)}$	output of the i-th period ($y_{(i)} = y(t),\quad t \in [t^0_{(i)}, t^L_{(i)}]$)
$x^0_{(i)}$	initial state of the i-th period
$x^L_{(i)}$	terminal state of the i-th period
b	error bound parameter (defined in (8))
$\Delta t_{(i)}$	excess converging time (defined in (9))

an error bound parameter b with an excess converging time $\Delta t_{(i)}$ is introduced, so the initial state of ILC in each period is sufficiently close to the stationary point $x(0)$. However, if $\Delta t_{(i)}$ converges on 0 as i grows, then we obtain the desired periodic trajectory asymptotically. The following section discusses the behavior of the excess converging time $\Delta t_{(i)}$.

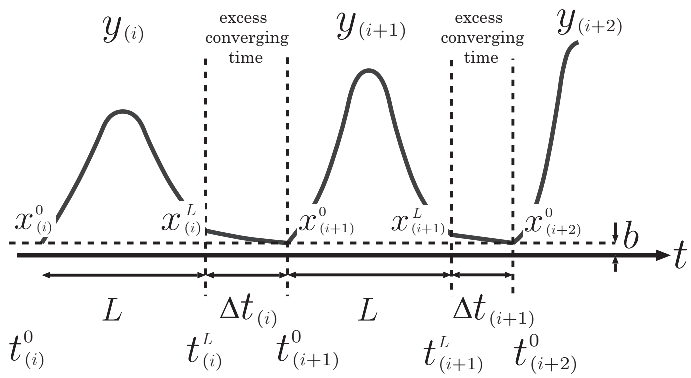

Figure 1: Typical time response of the proposed repetitive control procedure

3.2. Convergence analysis

In the procedure proposed in the previous section, we have introduced an error bound parameter b in (8) and, consequently, we need to employ the corresponding excess converging time $\Delta t_{(i)}$, which is 0 in the conventional repetitive control. This section discusses when the converging time $\Delta t_{(i)}$ becomes (converges on) 0.

The plant system considered here is a Hamiltonian system (1) with the following additional assumptions. Here, we use notation below to describe the plant system (1) for simplicity:

$$\dot{x} = f(x, u). \tag{11}$$

Assumption 2. A system of the form (1) has an equilibrium point at the origin; that is, $f(0, 0) = 0$ holds for any $t \geq 0$. Suppose also that the system is input-to-state stable.

In what follows, we suppose that each control input u for system (1) is generated by the iteration law (7); that is, we only consider the learning control input as u. We assume the following.

Assumption 3. Under an appropriately chosen learning gain $K_{(i)}$ in the iteration law (7), there exists a positive constant k_1 such that

$$\sup_{\substack{i \\ t^0 \leq t \leq t^0 + L}} \|u_{(i)}(t)\| \leq k_1$$

Due to our strategy mentioned in the previous subsection and Assumptions 2 and 3, for any $x(t^0)$ satisfying $\|x(t^0)\| < b$ with the error bound parameter b in (8) and any $t^0 \geq 0$, there exists a positive constant $k_2(b)$ such that

$$\begin{aligned}\|x(t^0 + L)\| &\leq \omega(\|x(t^0)\|, L) + \gamma\left(\sup_{\substack{i\\ t^0 \leq t \leq t^0+L}} \|u_{(i)}(t)\|\right)\\ &< \omega(b, L) + \gamma(k_1) =: k_2(b),\end{aligned}$$

where ω and γ represent class $\mathcal{KL}$ and class $\mathcal{K}$ functions [12], respectively. We utilize the notation B_r as an open ball with radius r whose center is at the origin.

Assumption 4. The origin of the system with $u \equiv 0$, that is, $f(0, 0)$ is locally exponentially stable for all $x(t^0) \in B_{k_2(b)}$.

For a feedback control design for a Hamiltonian system, a generalized canonical transformation is proposed in [11]. It is a pair of feedback and coordinate transformations preserving the Hamiltonian structure in (1).

Assumption 5. For any $t \in [t^0, t^0 + L]$ and any bounded input u, there exists a positive constant $k_3(u)$, and $f(x, u)$ in (1) satisfies the following inequality for any $x_n(t^0), x_m(t^0) \in B_b$:

$$\|f(x_n(t), u(t)) - f(x_m(t), u(t))\| \leq k_3(u)\|x_n(t) - x_m(t)\|.$$

Assumption 6. The Jacobian matrix $\frac{\partial f(0,0)}{\partial x}$ is bounded on $B_{k_2(b)}$.

Remark 2. From Assumptions 2, 4, 5 and 6, the origin of the following variational system

$$\dot{x}_v = \frac{\partial f(x, 0)}{\partial x} x_v \tag{12}$$

along any trajectory x of system (1) with $u \equiv 0$ starting from $x(t^0) \in B_{k_2(b)}$ is locally exponentially stable.

Since we prove a convergence theorem of the proposed method by utilizing the contraction mapping theorem [12], we introduce the following mappings (Fig. 1 may help in understanding them).

Definition 1. A mapping g_1 is defined as $g_1 : x^0_{(i)} \mapsto x^L_{(i)}$, which maps from the initial state for the i-th iteration to the terminal one after time period L along (1).

Definition 2. A mapping g_2 is defined as $g_2 : x^L_{(i)} \mapsto x^0_{(i+1)}$, which maps from the terminal state for the i-th iteration to the initial one for the i+1-th iteration after the excess converging time $\Delta t_{(i)}$ in (9) with $u(t) = 0$ for $\forall t \in [t^L_{(i)}, t^L_{(i)} + \Delta t_{(i)}]$.

We give the following lemma, which will be utilized to prove our convergence theorem.

Lemma 2. *Suppose that we use the same control input u for each iteration interval $t \in [t^0_{(i)}, t^L_{(i)}]$. Then there exists a positive constant b_{max} such that, if the error bound parameter b in (8) satisfies $b \leq b_{max}$, then the mapping*

$$g_2 \circ g_1 \ : \ x^0_{(i)} \mapsto x^0_{(i+1)} \tag{13}$$

is a contraction mapping.

Proof. Let ξ, η be two state trajectories of system (11) starting from $\xi(t^0) = \xi^0 \in B_b$ and $\eta(t^0) = \eta^0 \in B_b$, respectively. From Assumption 5, for $\forall t \in [t^0, t^0 + L]$, we have

$$\begin{aligned}\|\xi(t) - \eta(t)\| &\leq \|\xi^0 - \eta^0\| + \int_{t^0}^{t} \|f(\xi(\tau), u(\tau)) - f(\eta(\tau), u(\tau))\| \,\mathrm{d}\tau \\ &\leq \|\xi^0 - \eta^0\| + \int_{t^0}^{t} k_3(u)\|\xi(\tau) - \eta(\tau)\| \,\mathrm{d}\tau.\end{aligned}$$

The Gronwall-Bellman lemma [12] yields

$$\begin{aligned}\|\xi(t) - \eta(t)\| &\leq \|\xi^0 - \eta^0\| + \int_{t^0}^{t} \|\xi^0 - \eta^0\| k_3(u) \exp\Big[\int_{s}^{t} k_3(u) \ \mathrm{d}\tau\Big] \ \mathrm{d}s \\ &\leq \|\xi^0 - \eta^0\| \exp\Big[\int_{t^0}^{t} k_3(u) \ \mathrm{d}\tau\Big].\end{aligned} \tag{14}$$

By considering the time interval for the i-th iteration $t^0_{(i)} \leq t \leq t^0_{(i)} + L$, inequality (14) implies that

$$\|\xi^L_{(i)} - \eta^L_{(i)}\| \leq \|\xi^0_{(i)} - \eta^0_{(i)}\| \exp\Big[\int_{t^0}^{t^0+L} k_3(u) \ \mathrm{d}\tau\Big]. \tag{15}$$

From Definition 2, we can write

$$\|\xi^0_{(i+1)} - \eta^0_{(i+1)}\| = \|g_2(\xi^L_{(i)}) - g_2(\eta^L_{(i)})\|. \tag{16}$$

Next, we define the following term:

$$G(\eta^L_{(i)}, v_{(i)}) := g_2(\eta^L_{(i)} + v_{(i)}) - g_2(\eta^L_{(i)}), \tag{17}$$

where $v_{(i)} := \xi^L_{(i)} - \eta^L_{(i)}$. From definition (17), $G(\eta^L_{(i)}, 0) = 0$. Then, by applying the Morse lemma [13] to Eq.(17), we obtain

$$G(\eta^L_{(i)}, v_{(i)}) = \Big(\int_0^1 \frac{\partial G(\eta^L_{(i)}, w)}{\partial w}\Big|_{w=\lambda v_{(i)}} \mathrm{d}\lambda \Big) v_{(i)} =: \Phi(\eta^L_{(i)}, v_{(i)}) v_{(i)}. \tag{18}$$

Equations (17) and (18) lead to $g_2(\xi^L_{(i)}) - g_2(\eta^L_{(i)}) = \Phi(\eta^L_{(i)}, v_{(i)})(\xi^L_{(i)} - \eta^L_{(i)})$, and we can calculate that

$$\begin{aligned}
\|g_2(\xi^L_{(i)}) - g_2(\eta^L_{(i)})\| &\le \Big\|\Phi(\eta^L_{(i)}, v_{(i)})\Big\| \|\xi^L_{(i)} - \eta^L_{(i)}\| \\
&\le \|\xi^L_{(i)} - \eta^L_{(i)}\| \int_0^1 \Big\| \frac{\partial G(\eta^L_{(i)}, w)}{\partial w}\Big|_{w=\lambda v_{(i)}} \Big\| \mathrm{d}\lambda \\
&\le \|\xi^L_{(i)} - \eta^L_{(i)}\| \sup_{0\le\lambda\le 1} \Big\| \frac{\partial (g_2(\eta^L_{(i)} + w) - g_2(\eta^L_{(i)}))}{\partial w}\Big|_{w=\lambda v_{(i)}} \Big\| \\
&= \|\xi^L_{(i)} - \eta^L_{(i)}\| \sup_{0\le\lambda\le 1} \Big\| \frac{\partial g_2(\eta^L_{(i)} + w)}{\partial w}\Big|_{w=\lambda v_{(i)}} \Big\| \\
&\le \|\xi^L_{(i)} - \eta^L_{(i)}\| \sup_{\eta \in B_{k_2(b)}} \Big\| \frac{\partial g_2(\eta)}{\partial \eta} \Big\|.
\end{aligned} \tag{19}$$

Here we consider the variational system of (12) along a trajectory η starting from $\forall \eta^L \in B_{k_2(b)}$ with $u \equiv 0$. From Remark 2, for each η, there exist positive constants $\alpha(\eta), \beta(\eta)$, and $c(\eta)$ satisfying

$$\|x_v(t^0 + \Delta t(b))\| \le \beta(\eta) \|x_v(t^0)\| e^{-\alpha(\eta)\Delta t(b)}, \quad \text{for } \forall x_v(t^0) \in B_{c(\eta)}. \tag{20}$$

Note that we describe the excess converging time as $\Delta t(b)$ in (20) in order to clarify that it depends on the error bound parameter b. By using inequality (20) and from Definition 2, the following equation holds:

$$\sup_{\eta \in B_{k_2(b)}} \Big\| \frac{\partial g_2(\eta)}{\partial \eta} \Big\| = \sup_{\substack{\eta \in B_{k_2(b)} \\ x_v(t^0) \in B_{c(\eta)}}} \frac{\|x_v(t^0 + \Delta t(b))\|}{\|x_v(t^0)\|} \le \bar{\beta}(b) e^{-\bar{\alpha}(b)\Delta t(b)}, \tag{21}$$

where $\bar{\alpha}(b) := \sup_{\eta \in B_{k_2(b)}} \alpha(\eta)$ and $\bar{\beta}(b) := \sup_{\eta \in B_{k_2(b)}} \beta(\eta)$, respectively. From Eqs.(15),(16),(19) and (21), we have

$$\|\xi^0_{(i+1)} - \eta^0_{(i+1)}\| \leq M(b)\|\xi^0_{(i)} - \eta^0_{(i)}\|,$$

where

$$M(b) := \bar{\beta}(b) e^{-\bar{\alpha}(b)\Delta t(b)} \exp\left[\int_{t^0}^{t^0+L} k_3(u)\ \mathrm{d}\tau\right]. \tag{22}$$

Since the excess converging time $\Delta t(b)$ represents a time interval until the state asymptotically reaches B_b under the zero control input, and both $\bar{\alpha}(b)$ and $\bar{\beta}(b)$ have positive lower limits under $b \to 0$ from their definitions, one can let $\Delta t(b)$ be arbitrarily large by letting b be small. Therefore, there exists a positive constant b_{max} such that, if $b \leq b_{max}$, then $M(b)$ in (22) is less than 1. This implies that the mapping $g_2 \circ g_1$ becomes a contraction mapping. This proves the lemma. ■

Theorem 3. *There exists a constant $b_{\max} > 0$ such that, for any positive $b \leq b_{\max}$, there exists $\Delta t^b_\infty \geq 0$ satisfying*

$$\lim_{i \to \infty} \Delta t_{(i)} = \Delta t^b_\infty,$$

and the state $x(t)$ will converge on a $(L + \Delta t^b_\infty)$-periodic trajectory.

Proof. After a large enough period i satisfying $i > i_{stp}$, the assumptions of Lemma 2 hold. Lemma 2 and the contraction mapping theorem [12] imply that the mapping $g_2 \circ g_1$ in (13) has a unique fixed point $\bar{x}^0$ and that $\lim_{i\to\infty} x^0_{(i)} = \bar{x}^0$ holds. The fact that the initial state and the control input for each period converge as $i \to \infty$ and that the system is input-to-state stable from Assumption 2 prove the theorem. ■

Here, we can prove that the state trajectory always converges on a periodic trajectory. Furthermore, the following theorem guarantees that the excess converging time Δt^b_∞ also converges on 0 as b becomes smaller.

Theorem 4. *Suppose that iterative learning control applied to the plant achieves a trajectory tracking control*[3]. *Then the following equation holds:*

$$\lim_{b \to 0} \Delta t^b_\infty = 0$$

[3]For example, the authors' paper [5] proves that ILC for a trajectory tracking problem of simple mechanical systems always achieves the global minimum, i.e., the perfect tracking.

PROOF. Under $b \to 0$, the problem reduces to that of ILC mentioned in Section 2. From the assumption of the theorem, the control input converges to an optimal control input u^d which achieves an desired trajectory y^d, i.e.,

$$\lim_{b\to 0}\lim_{i\to\infty} \|u_{(i)} - u^d\|_{L_2} = 0. \tag{23}$$

Also, from the definition of b,

$$\lim_{b\to 0} \|x^0_{(i)}\| = 0. \tag{24}$$

Equations (23) and (24) and Assumption 2 yield

$$\lim_{b\to 0}\lim_{i\to\infty} \left(x_{(i)}{}^L - x^d(L)\right) = 0. \tag{25}$$

Equation (25) and the definition of b imply that

$$\lim_{b\to 0}\lim_{i\to\infty} \Delta t_{(i)} = 0.$$

This proves theorem.

This theorem implies that the proposed method lets the trajectory converge on the desired periodic one as the error bound parameter $b \to 0$. This fact allows us to obtain optimal control periodic solutions by a repetitive control framework.

4. Simulation

This section exhibits the effectiveness of the proposed method via numerical simulations. Here, let us consider a two-link robot manipulator moving on a horizontal plane, depicted in Fig. 2. As in the figure, the joint angles of the first and the second links are denoted by θ_1 and θ_2, respectively. The physical parameters of this apparatus are summarized in Table 2.

Then the dynamics of this apparatus is described by a Hamiltonian control system (1). Here, the generalized coordinate $q = (\theta_1, \theta_2)^\top \in \mathbb{R}^2$, and the generalized momentum p is described by $p = M(q)\dot{q}$, with the inertia matrix

$$M(q) = \begin{pmatrix} b_1 + b_2 + 2b_3\cos\theta_2 & b_2 + b_3\cos\theta_2 \\ b_2 + b_3\cos\theta_2 & b_2 \end{pmatrix},$$

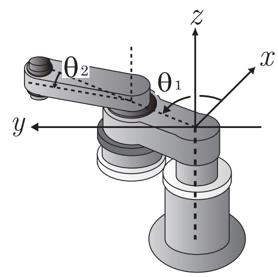

Figure 2: A two-link robot manipulator

Table 2: Physical parameters

θ_i	the joint angle of the i-th link	[rad]
m_i	the mas of the i-th link	[kg]
l_i	the length of the i-th link	[m]
l_{gi}	the length to the center of gravity	[m]
I_i	the inertia of the i-th link	[kgm^2]
d_i	the friction term of the i-th link	[Nms/rad]

where $b_1 := I_1+m_1l_{g2}^2+m_2l_1^2 = \frac{4}{3}m_1l_{g2}^2+m_2l_1^2$, $b_2 := I_2+m_2l_{g2}^2 = \frac{4}{3}m_2l_{g2}^2$ and $b_3 := l_1m_2l_{g2}$. The state $x := (q^\top, p^\top)^\top \in \mathbb{R}^4$, and the Hamiltonian is given by

$$H(x,u) = \frac{1}{2}p^{\mathrm{T}}M(q)^{-1}p - q^{\mathrm{T}}u.$$

The structure and dissipation matrices are given by

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{diag}(d_1, d_2) \end{pmatrix}.$$

The parameters used in the simulations are $b_1 = 2.292$, $b_2 = 0.600$, $b_3 = 0.750$, $d_1 = 0.2415$, and $d_2 = 0.2457$. See [14] for the details of this apparatus.

As explained in Section 3, we need to apply a local feedback designed by a generalized canonical transformation in order to obtain the closed-loop system as a Hamiltonian system. Here we employ a PD pre-feedback

$$u = \bar{u} - K_{\mathrm{P}}q - K_{\mathrm{D}}\dot{q} + \frac{\partial V(q)}{\partial q}^{\mathrm{T}}$$

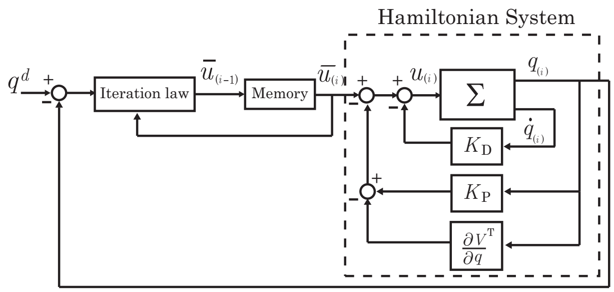

Figure 3: Closed-loop system with PD pre-feedback

with a new input $\bar{u}$ and the PD gains K_P and K_D, which are symmetric positive-definite matrices. The scalar function $V(q)$ denotes the virtual potential energy of the system. Then the dynamics of the closed-loop system is again described by a Hamiltonian system of the form (1), and this control system is depicted in Fig. 3. For this system, the output signal is $y = q = (\theta_1, \theta_2)^\top$, describing the joint angles.

4.1. Trajectory tracking control

Here we consider repetitive control for a trajectory tracking control problem. The desired trajectory for the output $y^d = (\theta_1^d, \theta_2^d)^\top$ is given by

$$\begin{aligned}
\theta_1^d(t) &:= \frac{1}{2}\sin\frac{\pi}{2}(t-1) + \frac{1}{2} \\
\theta_2^d(t) &:= \begin{cases} 0 & (0 \le t < \frac{3}{8}L) \\ \frac{1}{2}\sin\frac{\pi}{2}(t-1+\frac{3}{8}L) + \frac{1}{2} & (t \ge \frac{3}{8}L) \end{cases}
\end{aligned}$$

with the period $L = 4$ s, which is depicted in Fig. 4.

Now, apply the proposed method with the following cost function:

$$\Gamma(y) = \frac{1}{2}\|y - y^d\|_{L_2}^2. \tag{26}$$

Then the corresponding iterative learning law can be obtained from

$$\begin{aligned}
\bar{u}_{(2i+1)} &= \bar{u}_{(2i)} + \mathcal{R}(y_{(2i)} - y^d) \\
\bar{u}_{(2i+2)} &= \bar{u}_{(2i)} - K_{(i)}\mathcal{R}(y_{(2i+1)} - y_{(2i)}).
\end{aligned} \tag{27}$$

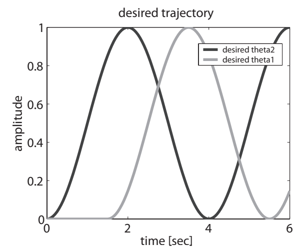

Figure 4: Desired trajectory

For the detailed derivation of Eq. (27), see [5, 6, 7]. The simulation results of the repetitive control are depicted in Figs. 5–8. Fig. 5 denotes the responses of the joint angles θ_1 and θ_2 from the 1st period to the 8th period. Fig. 6 denotes those with their reference signals in the 55th period. Both figures show that the joint angles are approaching their desired trajectories. Fig. 7 denotes the history of the cost function Γ in (26) with respect to the period i (the learning step). This figure shows that the joint angles converge on their reference trajectories, since the cost function decreases monotonically. Furthermore, Fig. 8 denotes the excess converging time $\Delta t_{(i)}$ with respect to the period i. This figure shows that the period $(L + \Delta t_{(i)})$ converges on 4 s after the 58 period. Thus the proposed repetitive control method works well with a trajectory tracking control problem.

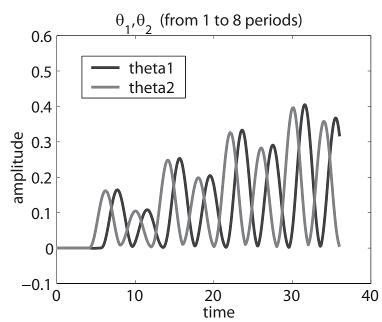

Figure 5: Responses of θ_1 and θ_2 (from the 1st period to the 8th period)

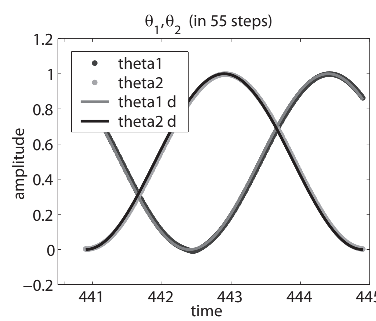

Figure 6: Responses of θ_1 and θ_2 (in the 55th period)

4.2. Optimal trajectory generation problem

Next, we consider repetitive control for an optimal control problem, that is, an optimal trajectory generation problem. As explained in Footnote 2,

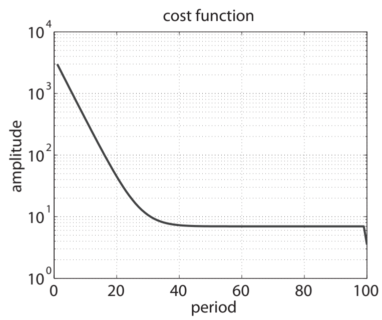

Figure 7: Cost function Γ for trajectory tracking control

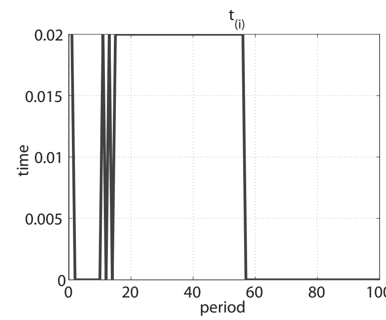

Figure 8: Excess converging time $\Delta t_{(i)}$ for trajectory tracking control

the desired (generated) trajectory has to be time symmetric with respect to the period L[4]. Therefore, we employ a special cost function which achieves desired intermediates and terminal joint angles, suppressing the input signal u and preserving the time symmetry of the trajectory

$$\Gamma(y) = \frac{1}{2}k_d\|F(t)(y-y^d)\|_{L_2}^2 + \frac{1}{2}k_t\|y-\mathcal{R}(y)\|_{L_2}^2 + \frac{1}{2}k_u\|u\|_{L_2}^2, \tag{28}$$

where k_d, k_t, and k_u are positive constants and the filter function $F(t)$ is defined with a small constant $\delta > 0$ as

$$F(t) := \begin{cases} \frac{1}{2}\left(1-\cos\left(\frac{-t_i+\delta+t}{\delta}\pi\right)\right) & (t_i-\delta \le t < t_i) \\ \frac{1}{2}\left(1-\cos\left(\frac{t_i+\delta-t}{\delta}\pi\right)\right) & (t_i \le t < t_i+\delta) \\ 0 & (\text{otherwise}) \end{cases},$$

where $t_i, i = 1, \ldots, n_p$ with a positive integer n_p, denotes the time for each desired intermediate (Fig. 9 illustrates $F(t)$ with $n_p = 1$ and $t_1 = L/2$). Here, the first term of the cost function Γ in (28) is for achieving the desired intermediates and terminal outputs (y^d is an appropriate 'virtual' time symmetric reference output which is only valid for t satisfying $F(t) \neq 0$), the second for time-symmetry, and the last for suppressing the input. Then the corresponding iterative control law is given as follows:

$$\begin{aligned} \bar{u}_{(2i+1)} &= \bar{u}_{(2i)} + \mathcal{R}\{k_dF(t)(y_{(2i)}-y^d) + 2k_t(y_{(2i)}-\mathcal{R}(y_{(2i)}))\} \\ \bar{u}_{(2i+2)} &= \bar{u}_{(2i)}(\mathrm{id}_U - k_uK_{(i)}) - K_{(i)}\mathcal{R}(y_{(2i+1)} - y_{(2i)}). \end{aligned}$$

[4]More precisely, the Hessian of the Hamiltonian function has to be so [5].

Let the period be $L = 4$ s, the initial state be $(0, 0)$, the desired terminal state be the same as the initial one, and the desired intermediate state be $y^d(L/2) = (\theta_1^d(L/2), \theta_2^d(L/2)) = (1.0, 1.0)$. The simulation results are depicted in Figs. 10–12. For reasons of space, only the responses with respect to the first joint are shown. Fig. 10 denotes the history of the cost function Γ in (28). The cost function decreases monotonically, and the desired trajectory is generated automatically. Fig. 11 denotes the time response of the angle of the first joint θ_1 in the 1st, 2nd, and 5th period, and every 10 steps from the 10th period to the 100th period. Furthermore, Fig. 12 denotes the history of $\Delta t_{(i)}$ with respect to period i. It eventually becomes quite small. Thus repetitive control for an optimal control problem also works well. These simulations demonstrate the effectiveness of the proposed algorithm.

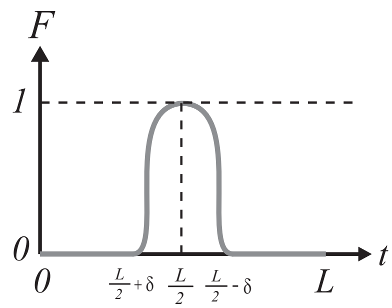

Figure 9: Illustration of the filter function $F(t)$ with $n_p = 1$ and $t_1 = L/2$

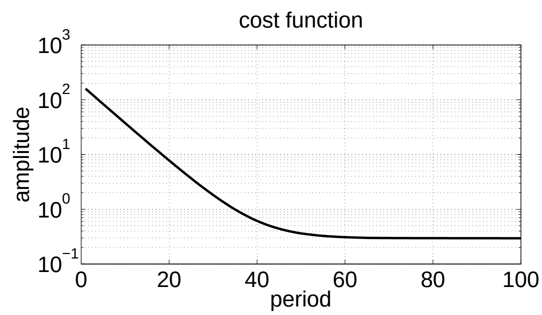

Figure 10: Cost function Γ for optimal control

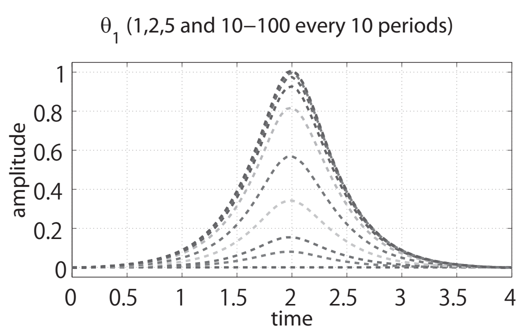

Figure 11: Response of θ_1 (in the 1st, 2nd, 5th, 10th, 20th, ... 100th periods)

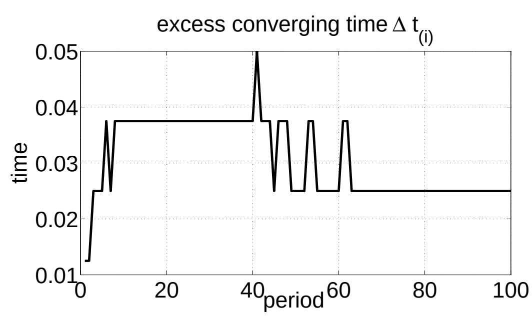

Figure 12: Excess converging time $\Delta t_{(i)}$ for optimal control

5. Conclusion

In this paper, we have proposed a new framework for repetitive control of Hamiltonian systems based on iterative learning control (ILC) proposed by the authors previously. Since this algorithm is based on the variational symmetry of Hamiltonian control systems, it is applicable to a class of infinite-dimensional optimal control problems as well as conventional trajectory tracking control problems.

We have shown the convergence analysis which guarantees the convergence of the trajectory generated by this procedure on the periodic reference or optimal trajectory under certain conditions. Furthermore, numerical simulations of a robot manipulator have shown the effectiveness of the proposed method. Since the main purpose of the paper is to propose our novel repetitive learning algorithm and to prove its convergence, quantitative analysis of performance of the proposed method is our future work.

So far, we have developed our ILC framework. We have executed laboratory experiments with a robot manipulator with input saturations in [6] and a nonholonomic vehicle systems in [15]. Further, we are tackling optimal gait generation for a walking robot, described as a nonlinear hybrid system in [16]. The method proposed here enables us to apply these techniques to a repetitive control framework as well as an ILC one. Thus the proposed method is expected to be useful for several purposes.

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