Passivity Based Control of Stochastic Port-Hamiltonian Systems

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***Abstract*—This paper introduces Stochastic Port-Hamiltonian Systems (SPHS's), whose dynamics are described by Itô stochastic differential equations. SPHS's are extension of the deterministic port-Hamiltonian systems which are used to express various passive systems. First, we show a necessary and sufficient condition to preserve the stochastic port-Hamiltonian structure of the system under a class of coordinate transformations. Second, we derive a condition for the system to be stochastic passive. Third, we equip Stochastic Generalized Canonical Transformations (SGCT's), which are pairs of coordinate and feedback transformations preserving the stochastic port-Hamiltonian structure. Finally, we propose a stochastic stabilization framework based on stochastic passivity and SGCT's.**

***Index Terms*—Nonlinear stochastic control, Stochastic Hamiltonian systems, Stochastic passivity, Stochastic stability.**

I. INTRODUCTION

There exist various disturbances such as measurement noises, modeling errors and so on in controlling real plants. Since they sometimes cause performance degradation or destabilization of the control system, it is important to consider them. Stochastic control theory is one of the efficient control methods which can take such disturbances into account. Theories and techniques for the deterministic dynamical systems are applied to stochastic ones described by stochastic differential equations [1]. Conserved quantities and symmetry for the stochastic systems are formulated in [2]. Lyapunov function approaches to stochastic stability are introduced in [3], [4]. Nonnegative supermartingales are used as stochastic Lyapunov functions, and asymptotic convergence of sample trajectories is proven by the martingale convergence theorem (see also [5]). The notion of stochastic passivity is introduced in [6]. As the deterministic passivity-based control [7], asymptotic stability in probability can be achieved for a stochastic nonlinear system by the unity feedback of a passive output. For other useful stabilization methods, the literature [8] deals with a stochastic version of a control Lyapunov function approach for a class of input-affine nonlinear stochastic systems. It provides a sufficient condition for asymptotic stabilizability in probability. In [9], a stabilization method based on the maximal Lyapunov exponent and the stochastic averaging is proposed. It provides a necessary and sufficient condition for asymptotic stability in probability of the original system as the maximal Lyapunov exponent of a reduced order averaged system becomes negative.

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The aim of this paper is to introduce Stochastic Port-Hamiltonian Systems (SPHS's). They are extension of deterministic port-Hamiltonian systems [10], [11], which are input-affine Hamiltonian systems. Although they intrinsically have useful properties for control such as invariance and passivity, SPHS's do not always possess similar properties. So, we clarify the corresponding properties of a SPHS and propose a stabilization method based on them. The results of the paper grew out of our previous reports [12], [13]. In this paper, we first define a SPHS and show a necessary and sufficient condition for the SPHS structure to be preserved under a class of coordinate transformations. Second, we derive a condition for the system to be stochastic passive. Third, we introduce Stochastic Generalized Canonical Transformation (SGCT) which is an extension of the generalized canonical transformation for deterministic port-Hamiltonian systems [14]. This transformation consists of a pair of coordinate and feedback transformations preserving the SPHS structure. Finally, we propose a stabilization method based on stochastic passivity and SGCT's, and provide an optimality condition for the proposed controller. This method is summarized as follows: we transform a SPHS to another stochastic passive one by a SGCT, and then stabilize the transformed system by the output feedback based on stochastic passivity. Here we compare the proposed method to other existing ones. Since we utilize stochastic passivity, and we can show a condition under which a transformed Hamiltonian becomes a stochastic control Lyapunov function, our method can be a special class of [6], [8]. However, we provide a systematic procedure based on the SGCT for possessing passivity and for constructing a Lyapunov function. Passivity of a system and existence of an appropriate Lyapunov function are pre-assumed in [6], [8], respectively. Besides, compared to [9], we only provide a sufficient condition for asymptotic stability in probability. However, our method is applicable to a wider class of the plant systems, since the literature [9] has particular restrictions due to the averaging. Although we provide explicit conditions for constructing a stabilizing controller, it is difficult to analytically obtain the maximal Lyapunov exponent, and numerical computations are necessary for designing their controller.

The present paper is also motivated by the authors' former study, where we proposed a learning optimal control scheme based on a symmetric property of deterministic Hamiltonian systems [15], [16], [17]. This learning method allows one to obtain an optimal feedforward control input minimizing a cost function by iteration of laboratory experiments without using precise knowledge of the plant model. However, not only this method but also many conventional iterative learning control methods, e.g., [18] do not consider any environmental disturbance and measurement noise during experiments. To solve this problem, we consider the plant system with the above uncertainties as a stochastic system, i.e., a SPHS and we focus on stochastic control theory. Since the proposed SPHS's can represent a wide class of important systems in the presence of noise such as mechanical systems, electro-mechanical systems, nonholonomic systems and so on, the results here can be applied not only to an extension of the learning control method but also to general nonlinear stochastic control problems.

The remainder of the paper is organized as follows. In section II, a SPHS is defined and conditions for invariance and stochastic passivity of the system are shown. In section III, a SGCT is introduced. In section IV, a stabilization method based on stochastic passivity and SGCT's is proposed. Finally, in section V, numerical examples demonstrate the effectiveness of the proposed method. In the sequel, for a twice differentiable scalar function $H(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, we define $\frac{\partial H(x,t)}{\partial x}$ as a n-dimensional row vector such that its i-th column is given by $\frac{\partial H(x,t)}{\partial x^i}$, where the superscript $(\cdot)^i$ denotes the i-th row of $(\cdot)$. We also define $\frac{\partial}{\partial x}\left(\frac{\partial H(x,t)}{\partial x}\right)^\top$ as a $n \times n$ matrix such that its (i,j)-th element is given by $\frac{\partial^2 H(x,t)}{\partial x^j \partial x^i}$, and $\frac{\partial}{\partial t}\left(\frac{\partial H(x,t)}{\partial x}\right)^\top$ as a n-dimensional column vector such that its i-th row is given by $\frac{\partial^2 H(x,t)}{\partial t \partial x^i}$.

II. Some properties of stochastic port-Hamiltonian systems

This section proposes Stochastic Port-Hamiltonian Systems (SPHS's) which are extension of port-Hamiltonian systems [10], [11], and clarifies some of their properties.

A. Stochastic Port-Hamiltonian Systems (SPHS's)

A port-Hamiltonian system is described by the following form of (1) and it is utilized to express various deterministic passive systems:

$$\begin{cases} \dot{x} = (J(x,t) - R(x,t))\dfrac{\partial H(x,t)}{\partial x}^\top + g(x,t)u \\ y = g(x,t)^\top \dfrac{\partial H(x,t)}{\partial x}^\top \end{cases}. \quad (1)$$

We extend this system to a stochastic dynamical system which is described by the following Itô stochastic differential equation:

$$\begin{cases} \mathrm{d}x = (J(x,t) - R(x,t))\dfrac{\partial H(x,t)}{\partial x}^\top \mathrm{d}t \\ \qquad +g(x,t)u\,\mathrm{d}t + h(x,t)\,\mathrm{d}w \\ y \ = g(x,t)^\top \dfrac{\partial H(x,t)}{\partial x}^\top \end{cases}. \quad (2)$$

Here $x(t) \in \mathbb{R}^n, u(t), y(t) \in \mathbb{R}^m$ describe the state, the input and the output, respectively. In this paper, although $x, u,$ and y basically represent functions of time and $x(t), u(t)$ and $y(t)$ represent their values evaluated for a fixed t, we sometimes drop "(t)" for notational simplicity.

The structure matrix $J(x,t) \in \mathbb{R}^{n\times n}$ and the dissipation matrix $R(x,t) \in \mathbb{R}^{n\times n}$ are skew-symmetric and symmetric positive semi-definite for all x and $t \ge 0$, respectively. The matrix $R(x,t)$ represents dissipative elements such as friction of mechanical systems and resistance of electric circuits. Matrix valued functions $g(x,t) \in \mathbb{R}^{n\times m}$ and $h(x,t) \in \mathbb{R}^{n\times r}$ represent the control and the noise ports, respectively. Here we suppose that $h(0,t) = 0_{n\times r}$ for all $t \ge 0$, where $0_{i\times j}$ denotes the $i \times j$ zero matrix. The signal $w(t) \in \mathbb{R}^r$ is a standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is a sample space, $\mathcal{F}$ is the sigma algebra of the observable random events and $\mathcal{P}$ is a probability measure on Ω. We suppose that a Hamiltonian $H(x,t) \in \mathbb{R}$, which describes the total energy of the system, is a sufficiently smooth function, and that the input u is a $\mathbb{R}^m$-valued measurable function and satisfies $E[\int_0^t \|u(s)\|^2\,\mathrm{d}s] < \infty$ with the expectation with respect to the measure $\mathcal{P}$, denoted by $E[\cdot]$. By setting $f_H(x,u,t) := (J-R)\frac{\partial H}{\partial x}^\top + gu$, we also suppose that there exist positive constants k_{L1}, k_{L2} and k_G such that the following inequalities hold for all $\xi(t), \eta(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m$ and $t \ge 0$:

$$\begin{aligned} &\|f_H(\xi,u,t) - f_H(\eta,u,t)\| \le k_{L1}\|\xi - \eta\| \\ &\|h(\xi,t) - h(\eta,t)\| \le k_{L2}\|\xi - \eta\| \\ &\|f_H(\xi,u,t)\| + \|h(\xi,t)\| \le k_G(1 + \|\xi\| + \|u\|). \end{aligned}$$

We define a stochastic system described by the form of (2) as a **Stochastic Port-Hamiltonian System (SPHS)**.

Remark 1: Suppose that the noise port does not exist, i.e., $h(x,t) \equiv 0_{n\times r}$ for any x and t. Then SPHS reduces to the conventional deterministic port-Hamiltonian system (1) as a special case.

In this paper, according to [4], [19], [6], [20], we introduce the notion of stability in probability for the system (2) as follows.

Definition 1: The origin of the system (2) is stable in probability if and only if for any $\epsilon > 0$ and $\delta > 0$, there exists $d(\epsilon,\delta) > 0$ such that if the initial condition $x(0)$ satisfies $\|x(0)\| < d(\epsilon,\delta)$, then $\mathcal{P}\left\{\sup_{t\ge 0}\|x(t)\| > \epsilon\right\} < \delta$ holds.

Definition 2: The origin of the system (2) is asymptotically stable in probability if and only if it is stable in probability and for any $\epsilon > 0$, $\lim_{T\to\infty}\mathcal{P}\left\{\sup_{t\ge T}\|x(t)\| > \epsilon\right\} = 0$ holds.

B. Coordinate transformations preserving SPHS structure

First, we reveal a property of SPHS's with respect to coordinate transformations. We consider the following time-varying coordinate transformation

$$\begin{cases} \bar{t} &= t \\ \bar{x} &= \Phi(x,t) \end{cases}, \quad (3)$$

such that for any t, a map $\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ defined as $\Phi_t(x) := \Phi(x,t) = \bar{x}$ is invertible. In the sequel, we utilize the notation $\Phi^{-1}(\bar{x},\bar{t})$ as $\Phi_{\bar{t}}^{-1}(\bar{x})$, that is, $\Phi^{-1}(\bar{x},\bar{t}) = \Phi_t^{-1}(\bar{x}) = x$. The following lemma is obtained in [14] for the deterministic port-Hamiltonian system in (1).

Lemma 1: [14] The deterministic port-Hamiltonian system in (1) is transformed into another one by the time-varying coordinate transformation (3), if and only if there exist a skew-symmetric matrix $K(x,t)$ and a symmetric matrix $S(x,t)$ such that $R(x,t)+S(x,t)$ is positive semi-definite and they satisfy

$$\frac{\partial \Phi(x,t)}{\partial x}(K(x,t)-S(x,t))\frac{\partial H(x,t)}{\partial x}^{\top}-\frac{\partial \Phi(x,t)}{\partial t}=0. \quad (4)$$

It follows that the system is always transformed into another one by any time-invariant coordinate transformation $\bar{x}=\Phi(x)$. Lemma 1 characterizes the class of coordinate transformations which preserves a deterministic port-Hamiltonian structure. However, the following lemma implies that the same class of transformations does not always preserve a SPHS structure:

Lemma 2: The stochastic port-Hamiltonian system (2) is transformed into another one by the time-varying coordinate transformation (3), if and only if there exist a skew-symmetric matrix $K(x,t)$ and a symmetric matrix $S(x,t)$ such that $R(x,t)+S(x,t)$ is positive semi-definite and they satisfy

$$\begin{aligned}&\frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i(x,t)}{\partial x}\right)^{\top} h(x,t)h(x,t)^{\top}\right\}\\ &\quad=\frac{\partial \Phi^i(x,t)}{\partial x}(K(x,t)-S(x,t))\frac{\partial H(x,t)}{\partial x}^{\top}-\frac{\partial \Phi^i(x,t)}{\partial t}\\ &\quad(i=1,2,\ldots,n). \end{aligned} \quad (5)$$

Proof: First, the necessity of Eq. (5) is shown. By utilizing Itô's formula [21], [22], the dynamics of the system in the new coordinate $\bar{x}^i,\ i=1,2,\ldots,n$ is calculated as

$$\begin{aligned}\mathrm{d}\bar{x}^i&=\frac{\partial \Phi^i}{\partial t}\,\mathrm{d}t+\frac{\partial \Phi^i}{\partial x}\,\mathrm{d}x+\frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\top} hh^{\top}\right\}\mathrm{d}t\\ &=\left[\frac{\partial \Phi^i}{\partial t}+\frac{\partial \Phi^i}{\partial x}(J-R)\frac{\partial H}{\partial x}^{\top}+\frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\top} hh^{\top}\right\}\right]\mathrm{d}t\\ &\quad+\frac{\partial \Phi^i}{\partial x}gu\,\mathrm{d}t+\frac{\partial \Phi^i}{\partial x}h\,\mathrm{d}w. \end{aligned} \quad (6)$$

Suppose that the system (2) is transformed into another one by the transformation. Then, the following equation holds for all u and w:

The Right Hand Side (R.H.S.) of Eq. (6)

$$\begin{aligned}&\equiv\left[(\bar{J}(\bar{x},\bar{t})-\bar{R}(\bar{x},\bar{t}))\frac{\partial H(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})}{\partial \bar{x}}^{\top}\right]^i\mathrm{d}\bar{t}\\ &\quad+\left[\bar{g}(\bar{x},\bar{t})u\right]^i\mathrm{d}\bar{t}+\left[\bar{h}(\bar{x},\bar{t})\,\mathrm{d}w\right]^i. \end{aligned} \quad (7)$$

Eq. (7) and that $\bar{t}$ is identical to t due to Eq. (3) imply that

$$\frac{\partial \Phi}{\partial x}g\equiv\bar{g}, \qquad \frac{\partial \Phi}{\partial x}h\equiv\bar{h}. \quad (8)$$

In what follows, the symbol $\cdot^{-1}$ represents the inverse function of the argument function and if no confusion arises, $[\cdot]^{-1}$ represents the inverse matrix of the argument. The first term of the right hand side of the identity (7) is calculated as follows:

$$\begin{aligned}&\left[(\bar{J}(\bar{x},\bar{t})-\bar{R}(\bar{x},\bar{t}))\frac{\partial H(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})}{\partial \bar{x}}^{\top}\right]^i\\ &=\left[\frac{\partial \Phi}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}(\bar{J}-\bar{R})\left[\frac{\partial \Phi}{\partial x}\right]^{-\top}\frac{\partial \Phi}{\partial x}^{\top}\frac{\partial H(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})}{\partial \bar{x}}^{\top}\right]^i\\ &=\frac{\partial \Phi^i}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}(\bar{J}-\bar{R})\left[\frac{\partial \Phi}{\partial x}\right]^{-\top}\frac{\partial H(x,t)}{\partial x}^{\top}. \end{aligned} \quad (9)$$

It follows from Eqs. (6), (7) and (9) that

$$\begin{aligned}&\frac{\partial \Phi^i}{\partial t}+\frac{\partial \Phi^i}{\partial x}(J-R)\frac{\partial H}{\partial x}^{\top}+\frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\top} hh^{\top}\right\}\\ &\qquad=\frac{\partial \Phi^i}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}(\bar{J}-\bar{R})\left[\frac{\partial \Phi}{\partial x}\right]^{-\top}\frac{\partial H}{\partial x}^{\top}. \end{aligned}$$

Consequently, we have

$$\begin{aligned}\frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\top} hh^{\top}\right\}&=\frac{\partial \Phi^i}{\partial x}\left[\left(\left[\frac{\partial \Phi}{\partial x}\right]^{-1}\bar{J}\left[\frac{\partial \Phi}{\partial x}\right]^{-\top}-J\right)\right.\\ &\quad\left.-\left(\left[\frac{\partial \Phi}{\partial x}\right]^{-1}\bar{R}\left[\frac{\partial \Phi}{\partial x}\right]^{-\top}-R\right)\right]\frac{\partial H^{\top}}{\partial x}-\frac{\partial \Phi^i}{\partial t}. \end{aligned} \quad (10)$$

From Eq. (10) we define the matrices $K(x,t)$ and $S(x,t)$ as

$$K(x,t):=\left[\frac{\partial \Phi}{\partial x}\right]^{-1}\bar{J}(\Phi(x,t),t)\left[\frac{\partial \Phi}{\partial x}\right]^{-\top}-J(x,t),$$

$$S(x,t):=\left[\frac{\partial \Phi}{\partial x}\right]^{-1}\bar{R}(\Phi(x,t),t)\left[\frac{\partial \Phi}{\partial x}\right]^{-\top}-R(x,t).$$

Then, $K(x,t)$ is skew-symmetric, since $J(x,t)$ and $\bar{J}(\Phi(x,t),t)$ are so. For $R(x,t)$ is symmetric and $\bar{R}(\Phi(x,t),t)$ is symmetric positive semi-definite, $S(x,t)$ is symmetric and $R(x,t)+S(x,t)$ is symmetric positive semi-definite.

The output $y(t)$ is well-defined, since

$$\begin{aligned}y&=\bar{g}(\bar{x},\bar{t})^{\top}\frac{\partial H(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})}{\partial \bar{x}}^{\top}\\ &=\left(\frac{\partial \Phi(x,t)}{\partial x}g(x,t)\right)^{\top}\left(\frac{\partial H(x,t)}{\partial x}\frac{\partial \Phi^{-1}(\bar{x},\bar{t})}{\partial \bar{x}}\right)^{\top}\\ &=\left(\frac{\partial \Phi(x,t)}{\partial x}g(x,t)\right)^{\top}\left(\frac{\partial H(x,t)}{\partial x}\left[\frac{\partial \Phi(x,t)}{\partial x}\right]^{-1}\right)^{\top}\\ &=g(x,t)^{\top}\frac{\partial H(x,t)}{\partial x}^{\top}. \end{aligned}$$

This proves the necessity of Eq. (5).

Second, the sufficiency of Eq. (5) is shown. The output y in the new coordinate $\bar{x}$ can be calculated by using Eq. (8) as

$$\begin{aligned}y&=g(x,t)^{\top}\left[\frac{\partial \Phi(x,t)}{\partial x}\right]^{\top}\left[\frac{\partial \Phi(x,t)}{\partial x}\right]^{-\top}\frac{\partial H(x,t)}{\partial x}^{\top}\\ &=\bar{g}(\bar{x},\bar{t})^{\top}\frac{\partial H(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})}{\partial \bar{x}}^{\top}. \end{aligned}$$

Therefore, the output y has the same form in the coordinate $\bar{x}$ as in x. Now suppose that Eq. (5) holds. Then, by utilizing

Eqs. (5) and (6), the dynamics of the system can be calculated in the new coordinate $\bar{x}$ as

$$\begin{aligned}
\mathrm{d}\bar{x}^i &= \left[\frac{\partial \Phi^i}{\partial x}(J-R)\frac{\partial H}{\partial x}^\top + \frac{\partial \Phi^i}{\partial x}(K-S)\frac{\partial H}{\partial x}^\top\right]\mathrm{d}t \\
&\quad + \frac{\partial \Phi^i}{\partial x} g u \,\mathrm{d}t + \frac{\partial \Phi^i}{\partial x} h \,\mathrm{d}w \\
&= \Bigg[\Bigg(\underbrace{\frac{\partial \Phi}{\partial x}(J+K)\frac{\partial \Phi}{\partial x}^\top}_{=:\,\bar{J}} - \underbrace{\frac{\partial \Phi}{\partial x}(R+S)\frac{\partial \Phi}{\partial x}^\top}_{=:\,\bar{R}}\Bigg) \\
&\quad \times \frac{\partial H(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})}{\partial \bar{x}}^\top\Bigg]^i \mathrm{d}t + \underbrace{\frac{\partial \Phi^i}{\partial x} g}_{=:\,\bar{g}^i} u \,\mathrm{d}t + \underbrace{\frac{\partial \Phi^i}{\partial x} h}_{=:\,\bar{h}^i} \mathrm{d}w \\
&= \left[(\bar{J}(\bar{x},\bar{t}) - \bar{R}(\bar{x},\bar{t}))\frac{\partial H(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})}{\partial \bar{x}}^\top\right]^i \mathrm{d}t \\
&\quad + \left[\bar{g}(\bar{x})u\right]^i \mathrm{d}\bar{t} + \left[\bar{h}(\bar{x})\,\mathrm{d}w\right]^i. \qquad (11)
\end{aligned}$$

Then, $\bar{J}(\bar{x},\bar{t})$ is skew-symmetric, since $J(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})$ and $K(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})$ are so. $\bar{R}(\bar{x},\bar{t})$ is symmetric positive semi-definite because of the assumption that $R(\Phi^{-1}(\bar{x},\bar{t}),\bar{t}) + S(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})$ is so. This proves the sufficiency of Eq. (5). ∎

Remark 2: Consider a deterministic port-Hamiltonian system (1) and apply Lemma 2 to the system. In this case, $h(x,t) \equiv 0_{n\times r}$ holds for all x and t. Then, Eq. (5) yields

$$\begin{aligned}
&\frac{\partial \Phi^i(x,t)}{\partial x}(K(x,t)-S(x,t))\frac{\partial H(x,t)}{\partial x}^\top - \frac{\partial \Phi^i(x,t)}{\partial t} = 0 \\
&(i = 1,2,\ldots,n). \qquad (12)
\end{aligned}$$

It implies the condition (4) in Lemma 1. Here suppose that Φ is a time-invariant coordinate transformation. Then, the condition (12) holds for any $H(x,t)$ and $\Phi(x)$ if we select the matrices $K(x,t) \equiv 0_{n\times n}$ and $S(x,t) \equiv 0_{n\times n}$, respectively. This implies that any time-invariant coordinate transformation always preserves the deterministic port-Hamiltonian structure. It shows that Lemma 2 implies the conventional result for the deterministic port-Hamiltonian system as a special case.

C. Stochastic passivity of SPHS's

In [14], a passivity condition was given, under which a deterministic port-Hamiltonian system (1) becomes passive. In this subsection, second, we investigate **stochastic passivity** of SPHS's (2). The notion of stochastic passivity was introduced in [6]. However, since the literature [6] deals with the property for only time-invariant stochastic systems, let us extend the concept to time-varying stochastic ones in a manner analogous to the deterministic time-varying case [14].

Definition 3: Consider the following nonlinear time-varying stochastic system:

$$\left\{\begin{array}{rcl} \mathrm{d}x &=& f(x,u,t)\,\mathrm{d}t + h(x,t)\,\mathrm{d}w \\ y &=& s(x,u,t) \end{array}\right. , \qquad (13)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n, h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times r}$ and $s : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ are sufficiently smooth functions and they satisfy $f(0,0,t) = 0_{n\times 1}$, $h(0,t) = 0_{n\times r}$ and $s(0,0,t) = 0_{m\times 1}$, respectively. Then the system (13) is said to be stochastic passive with respect to a storage function $V(x,t)$, if there exists a non-negative function $V(x,t) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that $V(x,t) \geq V(0,t) = 0$ and, for any $(x,u,t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$, it satisfies

$$\mathcal{L}V(x,t) \leq s(x,u,t)^\top u. \qquad (14)$$

Here $\mathcal{L}(\cdot)$ represents the infinitesimal generator for time-varying functions, defined by

$$\mathcal{L}(\cdot) := \frac{\partial(\cdot)}{\partial t} + \frac{\partial(\cdot)}{\partial x} f + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial(\cdot)}{\partial x}\right)^\top hh^\top\right\}. \qquad (15)$$

The following lemma characterizes stochastic passivity of SPHS's in (2).

Lemma 3: Consider the stochastic port-Hamiltonian system of the form (2). Suppose that a Hamiltonian $H(x,t)$ is a non-negative function such that $H(x,t) \geq H(0,t) = 0$. Then, the system is stochastic passive with respect to $H(x,t)$ if and only if the following inequality holds:

$$\begin{aligned}
&\frac{\partial H(x,t)}{\partial t} + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial H(x,t)}{\partial x}\right)^\top h(x,t)h(x,t)^\top\right\} \\
&\qquad \leq \frac{\partial H(x,t)}{\partial x} R(x,t) \frac{\partial H(x,t)}{\partial x}^\top. \qquad (16)
\end{aligned}$$

Proof: First, the necessity of the condition (16) is shown. According to the definition (15), $\mathcal{L}H(x,t)$ is calculated as

$$\begin{aligned}
&\mathcal{L}H(x,t) = \qquad (17) \\
&\frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\left((J-R)\frac{\partial H}{\partial x}^\top + gu\right) + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\right)^\top hh^\top\right\} \\
&= \frac{\partial H}{\partial t} - \frac{\partial H}{\partial x} R \frac{\partial H}{\partial x}^\top + y^\top u + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial H}{\partial x}\right)^\top hh^\top\right\}.
\end{aligned}$$

The last equality follows from $\frac{\partial H(x,t)}{\partial x} J(x,t) \frac{\partial H(x,t)}{\partial x}^\top = 0$ with the skew-symmetric matrix $J(x,t)$. Suppose that the system (2) is stochastic passive. Then, Eqs. (14) and (17) prove the necessity.

Second, the sufficiency is shown. From inequality (16) and Eq. (17), we obtain $\mathcal{L}H(x,t) \leq y^\top u$. According to the definition of stochastic passivity (14), this proves the sufficiency. ∎

Remark 3: Consider that we apply Lemma 3 to the deterministic port-Hamiltonian system (1) with the Hamiltonian $H(x,t)$ satisfying $H(x,t) \geq H(0,t) = 0$. For this system, $h(x,t) \equiv 0_{n\times r}$ holds. Then the passivity condition (16) reduces to

$$\frac{\partial H(x,t)}{\partial t} \leq \frac{\partial H(x,t)}{\partial x} R(x,t) \frac{\partial H(x,t)}{\partial x}^\top. \qquad (18)$$

The condition (18) reduces to Lemma 14 in [14] for the deterministic port-Hamiltonian system. Moreover, suppose that the system is time-invariant. Then the condition (18) always holds. It implies the result in [11] that any time-invariant deterministic port-Hamiltonian system with a positive definite Hamiltonian is passive. This shows that Lemma 3 implies the conventional results for deterministic port-Hamiltonian systems as a special case.

Remark 4: We discuss a physical interpretation of Lemma 3. First, we investigate the following two quantities with respect to the stochastic system (13) and a scalar function $V(x(t),t)$: $E[V(x(t+\mathrm{d}t),t+\mathrm{d}t)|x(t)]$ and $V(E[x(t+\mathrm{d}t)|x(t)],t+\mathrm{d}t)$. The first quantity is the expected value of V at the time $t+\mathrm{d}t$ conditioned on $x(t)$. According to Itô's formula, we have

$$E[V(x(t+\mathrm{d}t),t+\mathrm{d}t)|x(t)] = V(x(t),t) + \Bigg(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f + \frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)^{\top} h h^{\top}\right\}\Bigg)\,\mathrm{d}t + o(\mathrm{d}t), \tag{19}$$

where $o(\cdot)$ denotes the landau notation. Since $E[x(t+\mathrm{d}t)|x(t)] = x(t) + f(x,u,t)\,\mathrm{d}t$, the second quantity represents a realization of V at $t+\mathrm{d}t$ along the system without noise, and we have

$$\begin{aligned} &V(E[x(t+\mathrm{d}t)|x(t)],t+\mathrm{d}t)\\ &= V(x(t)+f(x,u,t)\,\mathrm{d}t + o(\mathrm{d}t), t+\mathrm{d}t)\\ &= V(x(t),t) + \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,u,t)\right)\mathrm{d}t + o(\mathrm{d}t). \end{aligned} \tag{20}$$

Since it follows from Eqs. (19) and (20), that

$$\begin{aligned} &E[V(x(t+\mathrm{d}t),t+\mathrm{d}t)|x(t)] - V(E[x(t+\mathrm{d}t)|x(t)],t+\mathrm{d}t)\\ &= \frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)^{\top} h h^{\top}\right\}\mathrm{d}t + o(\mathrm{d}t), \end{aligned}$$

the term $\frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)^{\top} h h^{\top}\right\}$ represents the expected variation of V caused by noise.

Consequently, we can interpret the inequality (16) in Lemma 3 as the expected energy increment due to the time-dependency of V and the noise effect should be dissipated by the friction in order to achieve stochastic passivity.

III. Recovery of Passivity via Stochastic Generalized Canonical Transformations (SGCT's)

Stochastic passivity is one of the useful concepts for stabilizing nonlinear stochastic systems. Since SPHS's are not always stochastic passive, as shown in Lemma 3, we propose a technique to render the system stochastic passive based on a special class of coordinate and feedback transformations called the generalized canonical transformations [14]. This technique plays an important role in our passivity based stochastic stabilization method proposed in the next section.

The generalized canonical transformations transform both the input and the output of the deterministic port-Hamiltonian system (1) into those of another port-Hamiltonian system with another Hamiltonian function. They are useful to design feedback controllers for deterministic Hamiltonian systems. This section, first, extends such transformations to stochastic versions for the SPHS (2). We clarify conditions for the transformations under which the transformed system preserves the SPHS structure. Second, we investigate an extra condition under which the transformed SPHS obtains stochastic passivity.

Let us define **Stochastic Generalized Canonical Transformations (SGCT's)**, which are natural extension of the deterministic generalized canonical transformations [14].

Definition 4: A set of transformations

$$\begin{aligned} \bar{t} &= t\\ \bar{x} &= \Phi(x,t)\\ \bar{H}(\bar{x},\bar{t}) &= H(x,t) + U(x,t)\ |_{x=\Phi^{-1}(\bar{x},\bar{t}),\ t=\bar{t}}\\ \bar{y} &= y + \alpha(x,t)\ |_{x=\Phi^{-1}(\bar{x},\bar{t}),\ t=\bar{t}}\\ \bar{u} &= u + \beta(x,t)\ |_{x=\Phi^{-1}(\bar{x},\bar{t}),\ t=\bar{t}} \end{aligned} \tag{21}$$

is said to be a stochastic generalized canonical transformation (SGCT) for the stochastic port-Hamiltonian system (2), if it transforms the system into another one which is also of the form (2) with the Hamiltonian $\bar{H}(\bar{x},\bar{t})$. Here the coordinate transformation $\bar{t}=t$ and $\bar{x}=\Phi(x,t)$ is the same as Eq. (3), and $U:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$, $\alpha:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^m$ and $\beta:\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^m$ are functions, respectively.

Now we show conditions which these transformations should satisfy.

Theorem 1: Consider the stochastic port-Hamiltonian system in (2). A set of transformations defined by (21) with the functions $\Phi(x,t)$, $U(x,t)$, $\alpha(x,t)$ and $\beta(x,t)$ yields a SGCT if and only if there exist a skew-symmetric matrix $P(x,t)$, a symmetric matrix $Q(x,t)$ such that $R(x,t)+Q(x,t)$ is positive semi-definite, and the functions $\Phi(x,t)$, $U(x,t)$ and $\beta(x,t)$ satisfy

$$\begin{aligned} \frac{\partial \Phi^i}{\partial t} + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\top} h h^{\top}\right\} = \frac{\partial \Phi^i}{\partial x}\Bigg[(J-R)\frac{\partial U}{\partial x}^{\top} + g\beta\\ + (P-Q)\frac{\partial (H+U)}{\partial x}^{\top}\Bigg],\quad (i=1,2,\dots,n). \end{aligned} \tag{22}$$

Further, a function $\alpha(x,t)$ is given by

$$\alpha(x,t) = g(x,t)^{\top}\frac{\partial U(x,t)}{\partial x}^{\top}. \tag{23}$$

Proof: First, the necessity of the theorem is shown. The dynamics of the transformed system in the new coordinate $(\bar{x},\bar{t})$ is calculated as (see, also Eq. (6))

$$\begin{aligned} \mathrm{d}\bar{x}^i = \left[\frac{\partial \Phi^i}{\partial t} + \frac{\partial \Phi^i}{\partial x}(J-R)\frac{\partial H}{\partial x}^{\top} + \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\top} h h^{\top}\right\}\right]\mathrm{d}t\\ + \frac{\partial \Phi^i}{\partial x} g u\,\mathrm{d}t + \frac{\partial \Phi^i}{\partial x} h\,\mathrm{d}w. \end{aligned} \tag{24}$$

Suppose that the SPHS in (2) is transformed into another one using a SGCT with Φ, U and β. Then, the following equation holds for all u and w

$$\begin{aligned} &\text{R.H.S. of Eq. (24)}\\ &\equiv \left[(\bar{J}-\bar{R})\frac{\partial \bar{H}(\bar{x},\bar{t})}{\partial \bar{x}}^{\top}\right]^i \mathrm{d}\bar{t} + \left[\bar{g}\bar{u}\right]^i \mathrm{d}\bar{t} + \left[\bar{h}\,\mathrm{d}w\right]^i\\ &= \frac{\partial \Phi^i}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}(\bar{J}-\bar{R})\left[\frac{\partial \Phi}{\partial x}\right]^{-\top}\frac{\partial (H(x,t)+U(x,t))}{\partial x}^{\top}\mathrm{d}t\\ &\quad + \left[\bar{g}(u+\beta)\right]^i \mathrm{d}t + \left[\bar{h}\,\mathrm{d}w\right]^i, \end{aligned} \tag{25}$$

where $\bar{J}(\bar{x},\bar{t}) \in \mathbb{R}^{n\times n}$ and $\bar{R}(\bar{x},\bar{t}) \in \mathbb{R}^{n\times n}$ are skew-symmetric and symmetric positive semi-definite matrices, respectively. Eq. (25) and that $\bar{t}$ is identical to t, imply that

$$\frac{\partial \Phi}{\partial x} g \equiv \bar{g}, \qquad \frac{\partial \Phi}{\partial x} h \equiv \bar{h}. \tag{26}$$

From Eqs. (24), (25) and (26), we have

$$\frac{\partial \Phi^i}{\partial t} + \frac{1}{2}\mathrm{tr}\left\{ \frac{\partial}{\partial x}\left(\frac{\partial \Phi^i}{\partial x}\right)^{\top} hh^{\top}\right\} = \frac{\partial \Phi^i}{\partial x}\left[\left[\frac{\partial \Phi}{\partial x}\right]^{-1}(\bar{J}-\bar{R}) \times \left[\frac{\partial \Phi}{\partial x}\right]^{-\top} \frac{\partial (H+U)}{\partial x}^{\top} - (J-R)\frac{\partial H}{\partial x}^{\top} + g\beta\right]. \tag{27}$$

Here we define the matrices $P(x,t)$ and $Q(x,t)$ as

$$\begin{aligned} P(x,t) &:= \left[\frac{\partial \Phi}{\partial x}\right]^{-1} \bar{J}(\Phi(x,t),t) \left[\frac{\partial \Phi}{\partial x}\right]^{-\top} - J(x,t), \\ Q(x,t) &:= \left[\frac{\partial \Phi}{\partial x}\right]^{-1} \bar{R}(\Phi(x,t),t) \left[\frac{\partial \Phi}{\partial x}\right]^{-\top} - R(x,t). \end{aligned} \tag{28}$$

Then, $P(x,t)$ is skew-symmetric, since $J(x,t)$ and $\bar{J}(\Phi(x,t),t)$ are so. For $R(x,t)$ is symmetric and $\bar{R}(\Phi(x,t),t)$ is symmetric positive semi-definite, $Q(x,t)$ is symmetric and $R(x,t) + Q(x,t)$ is symmetric positive semi-definite. By substituting Eq. (28) for Eq. (27), Eq. (22) is obtained immediately.

The change of the output $\alpha(x,t)$ which yields a SGCT (21) can be calculated as

$$\begin{aligned} \alpha &= \bar{y} - y = \bar{g}^{\top}\frac{\partial \bar{H}(\bar{x},\bar{t})}{\partial \bar{x}}^{\top} - g^{\top}\frac{\partial H(x,t)}{\partial x}^{\top} \\ &= g^{\top}\frac{\partial \Phi(x,t)}{\partial x}^{\top}\left[\frac{\partial \Phi(x,t)}{\partial x}\right]^{-\top}\frac{\partial (H(x,t)+U(x,t))}{\partial x}^{\top} \\ &\quad - g^{\top}\frac{\partial H(x,t)}{\partial x}^{\top} = g^{\top}\frac{\partial U(x,t)}{\partial x}^{\top}. \end{aligned}$$

This proves the necessity of the theorem.

Second, the sufficiency of the theorem is shown. Now suppose the assumption of the theorem holds. Then, by substituting Eq. (22) for Eq. (24), the dynamics of the system can be calculated in the new coordinate as

$$\begin{aligned} \mathrm{d}\bar{x}^i &= \left[\frac{\partial \Phi^i}{\partial x}(J-R)\frac{\partial H}{\partial x}^{\top} + \frac{\partial \Phi^i}{\partial x}\left[(J-R)\frac{\partial U}{\partial x}^{\top} + g\beta \right.\right. \\ &\quad \left.\left. + (P-Q)\frac{\partial (H+U)}{\partial x}^{\top}\right]\right]\mathrm{d}t + \frac{\partial \Phi^i}{\partial x} gu\,\mathrm{d}t + \frac{\partial \Phi^i}{\partial x} h\,\mathrm{d}w \\ &= \left[\frac{\partial \Phi}{\partial x}(J+P-R-Q)\frac{\partial \Phi}{\partial x}^{\top}\frac{\partial (H+U)}{\partial \bar{x}}^{\top}\right]^i \mathrm{d}\bar{t} \\ &\quad + \frac{\partial \Phi^i}{\partial x} g(u+\beta)\,\mathrm{d}\bar{t} + \frac{\partial \Phi^i}{\partial x} h\,\mathrm{d}w. \end{aligned} \tag{29}$$

Here, $\bar{J}(\bar{x},\bar{t}), \bar{R}(\bar{x},\bar{t}), \bar{g}(\bar{x},\bar{t})$ and $\bar{h}(\bar{x},\bar{t})$ are given by

$$\begin{aligned} \bar{J}(\bar{x},\bar{t}) &= \left.\frac{\partial \Phi(x,t)}{\partial x}(J(x,t)+P(x,t))\frac{\partial \Phi(x,t)}{\partial x}^{\top}\right|_{\substack{x=\Phi^{-1}(\bar{x},\bar{t}) \\ t=\bar{t}}} \\ \bar{R}(\bar{x},\bar{t}) &= \left.\frac{\partial \Phi(x,t)}{\partial x}(R(x,t)+Q(x,t))\frac{\partial \Phi(x,t)}{\partial x}^{\top}\right|_{\substack{x=\Phi^{-1}(\bar{x},\bar{t}) \\ t=\bar{t}}} \\ \bar{g}(\bar{x},\bar{t}) &= \left.\frac{\partial \Phi(x,t)}{\partial x}g(x,t)\right|_{\substack{x=\Phi^{-1}(\bar{x},\bar{t}) \\ t=\bar{t}}} \\ \bar{h}(\bar{x},\bar{t}) &= \left.\frac{\partial \Phi(x,t)}{\partial x}h(x,t)\right|_{\substack{x=\Phi^{-1}(\bar{x},\bar{t}) \\ t=\bar{t}}}. \end{aligned} \tag{30}$$

Then, $\bar{J}(\bar{x},\bar{t})$ is skew-symmetric, since $J(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})$ and $P(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})$ are so. $\bar{R}(\bar{x},\bar{t})$ is symmetric positive semi-definite because of the assumption that $R(\Phi^{-1}(\bar{x},\bar{t}),\bar{t}) + Q(\Phi^{-1}(\bar{x},\bar{t}),\bar{t})$ is so. Consequently, the dynamics of the system in the new coordinate $\bar{x}$ is given by

$$\begin{aligned} \mathrm{d}\bar{x}^i &= \left[\left(\bar{J}(\bar{x},\bar{t}) - \bar{R}(\bar{x},\bar{t})\right)\frac{\partial \bar{H}(\bar{x},\bar{t})}{\partial \bar{x}}^{\top}\right]^i \mathrm{d}\bar{t} + \left[\bar{g}(u+\beta)\right]^i \mathrm{d}\bar{t} \\ &\quad + \left[\bar{h}\,\mathrm{d}w\right]^i, \ (i=1,2,\ldots,n) \end{aligned} \tag{31}$$

The output in the new coordinate $\bar{y}$ is obtained by Eq. (23) as

$$\begin{aligned} \bar{y} &= y + \alpha(x,t) \\ &= g^{\top}\frac{\partial H(x,t)}{\partial x}^{\top} + g^{\top}\frac{\partial U(x,t)}{\partial x}^{\top} \\ &= g^{\top}\frac{\partial \Phi}{\partial x}^{\top}\left[\frac{\partial \Phi}{\partial x}\right]^{-\top}\frac{\partial (H+U)}{\partial x}^{\top} = \bar{g}^{\top}\frac{\partial \bar{H}(\bar{x},\bar{t})}{\partial \bar{x}}^{\top}. \end{aligned} \tag{32}$$

Eqs. (31) and (32) imply the sufficiency of the theorem. ■

Finally, the following lemma states a condition for a transformed SPHS by a SGCT to be stochastic passive.

Lemma 4: Consider the stochastic port-Hamiltonian system in (2) and transform it by an appropriate stochastic generalized canonical transformation such that $\bar{H}(\bar{x},\bar{t}) \geq \bar{H}(0,\bar{t}) = 0$. Then, the transformed system becomes stochastic passive with the new Hamiltonian $\bar{H}(\bar{x},\bar{t})$ as a storage function if and only if the following inequality holds:

$$\begin{aligned} &-\frac{\partial (H+U)}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}\frac{\partial \Phi}{\partial t} + \frac{\partial (H+U)}{\partial t} \\ &+ \frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial (H+U)}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}\right)^{\top} h(x)h(x)^{\top}\frac{\partial \Phi}{\partial x}^{\top}\right\} \\ &\leq \frac{\partial (H+U)}{\partial x}(R+Q)\frac{\partial (H+U)}{\partial x}^{\top}. \end{aligned} \tag{33}$$

Proof: Due to Lemma 3, the necessary and sufficient condition for stochastic passivity is that the following inequality holds in the new coordinate transformed by the SGCT:

$$\frac{\partial \bar{H}(\bar{x},\bar{t})}{\partial \bar{t}} + \frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{H}(\bar{x},\bar{t})}{\partial \bar{x}}\right)^{\top}\bar{h}\bar{h}^{\top}\right\} \leq \frac{\partial \bar{H}}{\partial \bar{x}}\bar{R}\frac{\partial \bar{H}}{\partial \bar{x}}^{\top}. \tag{34}$$

The first term on the left hand side of the inequality (34) is calculated as

$$\frac{\partial \bar{H}(\bar{x},\bar{t})}{\partial \bar{t}} = \frac{\partial (H+U)}{\partial x}\frac{\partial \Phi^{-1}(\bar{x},\bar{t})}{\partial \bar{t}} + \frac{\partial (H+U)}{\partial t}. \tag{35}$$

Since the Jacobian of the pair of the coordinate transformations $\bar{x} = \Phi(x,t)$ and $\bar{t} = t$, denoted by $\mathcal{J}$, is given by

$$\mathcal{J} = \begin{pmatrix} \dfrac{\partial \Phi(x,t)}{\partial x} & \dfrac{\partial \Phi(x,t)}{\partial t} \\ 0_{1\times n} & 1 \end{pmatrix},$$

the Jacobian of the inverse coordinate transformation which coincides with $\mathcal{J}^{-1}$ is obtained as

$$\begin{aligned} \mathcal{J}^{-1} &= \begin{pmatrix} \left[\dfrac{\partial \Phi(x,t)}{\partial x}\right]^{-1} & -\left[\dfrac{\partial \Phi(x,t)}{\partial x}\right]^{-1}\dfrac{\partial \Phi(x,t)}{\partial t} \\ 0_{1\times n} & 1 \end{pmatrix} \\ &\equiv \begin{pmatrix} \dfrac{\partial \Phi^{-1}(\bar{x},\bar{t})}{\partial \bar{x}} & \dfrac{\partial \Phi^{-1}(\bar{x},\bar{t})}{\partial \bar{t}} \\ 0_{1\times n} & 1 \end{pmatrix}. \end{aligned} \tag{36}$$

It follows from the identity (36) that

$$\frac{\partial \Phi^{-1}(\bar{x},\bar{t})}{\partial \bar{t}} = -\left[\frac{\partial \Phi(x,t)}{\partial x}\right]^{-1}\frac{\partial \Phi(x,t)}{\partial t}. \tag{37}$$

Here, the following equation holds:

$$\begin{aligned} \frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{H}}{\partial \bar{x}}\right)^{\top} &= \frac{\partial}{\partial x}\left(\frac{\partial (H+U)}{\partial x}\frac{\partial \Phi^{-1}}{\partial \bar{x}}\right)^{\top}\frac{\partial \Phi^{-1}}{\partial \bar{x}} \\ &= \frac{\partial}{\partial x}\left(\frac{\partial (H+U)}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}\right)^{\top}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}. \end{aligned} \tag{38}$$

Consequently, inequality (34) with Eqs. (30), (35), (37) and (38) imply that the condition (33) holds immediately. This proves the lemma. ∎

Example 1: Consider the following simple mechanical system with random noise:

$$\begin{cases} \begin{pmatrix} \mathrm{d}q \\ \mathrm{d}p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}\mathrm{d}t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u\,\mathrm{d}t + \begin{pmatrix} 0 \\ ap \end{pmatrix}\mathrm{d}w \\ y = \frac{\partial H}{\partial p} = \frac{p}{M} \end{cases} \tag{39}$$

with the Hamiltonian $H(x) = \frac{p^2}{2M}$, where $x := (q,p)^{\top} \in \mathbb{R}^2$ denotes the state, a positive constant M denotes the inertia and $w(t) \in \mathbb{R}$ is a standard Wiener process. Here we suppose that the noise port ap with $a > 0$. Since $\frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial H(x)}{\partial x}\right)^{\top} h(x)h(x)^{\top}\right\} = \frac{a^2p^2}{2M}$ and $\frac{\partial H(x)}{\partial x}R(x)\frac{\partial H(x)}{\partial x}^{\top} = 0$, it follows from Lemma 3 that this system is not stochastic passive. We assign any positive definite scalar function $U(q)$ so that $\bar{H}$ becomes positive definite. By using Theorem 1, we obtain a SGCT as $\bar{q} = q$, $\bar{p} = p$, $\alpha = 0$, $\beta = \frac{\mathrm{d}U(q)}{\mathrm{d}q} - \frac{Q_{22}}{M}p$, where $P := 0_{2\times 2}$ and $Q := \mathrm{diag}\{0, Q_{22}\}$. Then by using Lemma 4, the condition under which the transformed system becomes stochastic passive is $Ma^2 \le 2Q_{22}$.

Proposition 1: Suppose that the conditions in Lemma 4 are satisfied, and that $\bar{H}(\bar{x},\bar{t})$ is positive definite for all $\bar{t}$. Moreover, suppose that the inequality (34) holds strictly for all $\bar{x}\backslash\{0\}$ and $\bar{t}$, if $\frac{\partial \bar{H}(\bar{x},\bar{t})}{\partial \bar{x}}\bar{g}(\bar{x},\bar{t}) = 0$ holds for all $\bar{x}\backslash\{0\}$ and $\bar{t}$. Then, the transformed SPHS satisfies a stochastic Lyapunov condition proposed in [8] and $\bar{H}(\bar{x},\bar{t})$ is a control Lyapunov function.

Proof: According to the definition of the control Lyapunov function in [8], it is sufficient to show that if $\frac{\partial \bar{H}(\bar{x},\bar{t})}{\partial \bar{x}}\bar{g}(\bar{x},\bar{t}) = 0$ holds for all $\bar{x}\backslash\{0\}$ and $\bar{t}$, then $\mathcal{L}\bar{H}(\bar{x},\bar{t}) < 0$ holds.

Under the condition, we have

$$\mathcal{L}\bar{H}(\bar{x},\bar{t}) = \frac{\partial \bar{H}}{\partial \bar{t}} - \frac{\partial \bar{H}}{\partial \bar{x}}\bar{R}\frac{\partial \bar{H}}{\partial \bar{x}}^{\top} + \frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{H}}{\partial \bar{x}}\right)^{\top}\bar{h}\bar{h}^{\top}\right\}.$$

It follows from the assumptions in Proposition 1 that $\mathcal{L}\bar{H}(\bar{x},\bar{t}) < 0$. ∎

IV. Passivity based stochastic stabilization framework

This section proposes a stochastic stabilization framework for SPHS's based on stochastic passivity and the stochastic generalized canonical transformation (SGCT). For deterministic systems, LaSalle's invariance principle plays an important role in passivity based stabilization [7], [23]. This principle was originally applied to time-invariant systems. It is restricted in applications for time-varying ones with some difficulties, for example, the ω-limit set of a trajectory is itself not invariant. In the literatures [24], [6], stochastic versions of LaSalle's invariance principle were proposed for time-invariant stochastic systems. Then, in [19], a generalization for time-varying case was proposed. Since a stability result for time-varying stochastic systems is more conservative than that for time-invariant ones, we consider both cases separately. Finally, an optimality condition for the proposed controller is presented.

A. Stability theory for time-invariant SPHS's

By utilizing Lemmas 3 and 4, the following theorem states the proposed stabilization method.

Theorem 2: Consider a time-invariant stochastic port-Hamiltonian system of the form (2). Suppose that the system is transformed by an appropriate stochastic generalized canonical transformation such that $\bar{H}(\bar{x}) := H(\Phi^{-1}(\bar{x})) + U(\Phi^{-1}(\bar{x}))$ is positive definite and that $\bar{H}(\bar{x})$ is a d times differentiable function with $d \ge 1$. Then the state $\bar{x}$ of the transformed system tends in probability to the largest invariant set whose support is contained in the locus $\mathcal{L}\bar{H}(\bar{x}) = 0$ for any $t \ge 0$, with the unity feedback $\bar{u} = -\bar{y}$. Furthermore, if it holds that $\bar{\Gamma} \cap \bar{\Pi} = \{0\}$, then the unity feedback renders the system asymptotically stable in probability. Here, the distribution $\bar{\Lambda}$, the sets $\bar{\Gamma}$ and $\bar{\Pi}$ are defined as

$$\bar{\Lambda} = \mathrm{span}\{\mathrm{ad}_{\bar{f}_0}^k \bar{g}_i(\bar{x}) | 0 \le k \le n-1, 1 \le i \le m\}$$

$$\bar{\Gamma} = \{\bar{x} \in \mathbb{R}^n | \mathcal{L}_0^k \bar{H}(\bar{x}) = 0,\ k = 1, 2, \ldots, d\}$$

$$\bar{\Pi} = \{\bar{x} \in \mathbb{R}^n | \mathcal{L}_0^k L_\lambda \bar{H}(\bar{x}) = 0,\ \forall \lambda \in \bar{\Lambda},\ k = 0, 1, \ldots, d-1\},$$

where $\bar{f}_0 := (\bar{J}(\bar{x}) - \bar{R}(\bar{x}))\frac{\partial \bar{H}(\bar{x})}{\partial \bar{x}}^{\top}$, and $\bar{g}_i$ represents the i-th column of $\bar{g}$. The vector field $\mathrm{ad}_{\bar{f}_0}^k \bar{g}_i$ is defined as

$\mathrm{ad}_{\bar f_0}^k \bar g_i(\bar x) = [\bar f_0, \mathrm{ad}_{\bar f_0}^{k-1}\bar g_i](\bar x)$, for any integer $k \geq 1$, setting $\mathrm{ad}_{\bar f_0}^0 \bar g_i = \bar g_i$ with the Lie bracket $[\cdot,\cdot]$, see e.g. [25], [23]. $\mathcal{L}_0$ denotes the operator (15) in which f is replaced by $\bar f_0$ and $L_{(\cdot)}$ represents the Lie derivative along $(\cdot)$. $\mathcal{L}_0^k L_\lambda \bar H$ can be calculated as $\mathcal{L}_0^k L_\lambda \bar H(\bar x) = \mathcal{L}_0^k \frac{\partial \bar H}{\partial \bar x}\lambda = \mathcal{L}_0 \mathcal{L}_0^{k-1}\frac{\partial \bar H}{\partial \bar x}\lambda$ for any integer $k \geq 1$, setting $\mathcal{L}_0^0 L_\lambda \bar H = L_\lambda \bar H$.

Proof: Since the transformed system is stochastic passive with respect to $\bar H(\bar x)$, $\mathcal{L}\bar H(\bar x) \leq -\|\bar y\|^2 \leq 0$ holds for the closed loop system with the unity feedback $\bar u = -\bar y$. Then the stochastic version of LaSalle's theorem in [24], [6] implies that the state $\bar x$ tends in probability to the largest invariant set whose support is contained in the locus $\mathcal{L}\bar H(\bar x) = 0$ for any $t \geq 0$. Consequently, the rest of the theorem is shown by directly applying Corollary 4.7 in [6]. ■

B. Stability theory for time-varying SPHS's

We show the following stability theory for time-varying SPHS's. For the time-varying case, LaSalle's invariance principle in [19] generally guarantees the output convergence only.

Theorem 3: Consider a time-varying stochastic port-Hamiltonian system of the form (2). Suppose that the system is transformed by a stochastic generalized canonical transformation such that $\bar H(\bar x, \bar t)$ is positive definite for all $\bar t$ and $\frac{\partial \bar H}{\partial \bar x}$, $\frac{\partial}{\partial \bar x}\left(\frac{\partial \bar H}{\partial \bar x}\right)^\top$ and $\frac{\partial \bar H}{\partial \bar t}$ are continuous (matrix valued) functions, respectively. Furthermore, suppose that $\lim_{\|\bar x\|\to\infty} \inf_{0\leq \bar t<\infty} \bar H(\bar x,\bar t) = \infty$, and one of the following conditions holds:

(i) For each initial state $\bar x_0 \in \mathbb{R}^n$, there is a $d > 2$ such that $\sup_{0\leq \bar t<\infty} E[\|\bar x(\bar t)\|^d] < \infty$.
(ii) $\bar h(\bar x, \bar t)$ is bounded.
(iii) Almost every sample path of $\int_0^{\bar t} \bar h(\bar x(\tau),\tau)\,\mathrm{d}w$ is uniformly continuous on $\bar t \geq 0$.

Then, for every initial state $\bar x_0 \in \mathbb{R}^n$, the Hamiltonian function of the feedback system with the unity feedback $\bar u = -\bar y$ converges to a finite limit, that is, $\lim_{\bar t\to\infty} \bar H(\bar x,\bar t) < \infty$ exists almost surely. Furthermore,

$$\lim_{\bar t\to\infty} \bar y(\bar t) = 0_{m\times 1}$$

holds almost surely.

Proof: Since the transformed system is stochastic passive with respect to $\bar H(\bar x,\bar t)$, $\mathcal{L}\bar H(\bar x,\bar t) \leq -\|\bar y\|^2 \leq 0$ holds for the closed loop system with the unity feedback $\bar u = -\bar y$. Then the claim of the theorem is proven by directly applying Theorem 2.1 and Theorem 2.5 in [19]. ■

C. Optimality condition for the proposed controller

The following proposition shows an optimality condition for the proposed controller.

Proposition 2: Suppose that the conditions in Lemma 4 are satisfied. Moreover, suppose that $\bar H(\bar x,\bar t)$ is positive definite for all $\bar t$, and that the equality holds in the inequality (33) in Lemma 4. Then, the proposed controller $u = -\bar y - \beta = -g^\top \frac{\partial (H+U)}{\partial x}^\top - \beta$ is the optimal control for the following cost function $\Gamma(x,t)$ with any $T > 0$:

$$\Gamma(x,t) = E\left[\int_0^T \frac{1}{2}\frac{\partial (H+U)}{\partial x} g g^\top \frac{\partial (H+U)}{\partial x}^\top + \frac{1}{2}(u+\beta)^\top (u+\beta)\,\mathrm{d}t\right].$$

Furthermore, the value function is given by the new Hamiltonian $H(x,t) + U(x,t)$.

Proof: The Stochastic Hamilton-Jacobi-Bellman (SHJB) equation with the value function $H+U$ associated with this optimal control problem is expressed as

$$\begin{aligned}-\frac{\partial (H+U)}{\partial t} = \min_u \Bigg(&\frac{1}{2}\frac{\partial (H+U)}{\partial x} g g^\top \frac{\partial (H+U)}{\partial x}^\top \\ &+\frac{1}{2}(u+\beta)^\top(u+\beta) + \frac{\partial (H+U)}{\partial x}\left((J-R)\frac{\partial H}{\partial x}^\top + g u\right) \\ &+\frac{1}{2}\operatorname{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial (H+U)}{\partial x}\right)^\top h h^\top\right\}\Bigg). \end{aligned} \tag{40}$$

By taking the gradient with respect to u of the inside of the parenthesis on the right hand side, and setting it to zero, the optimal control is given by $u = -g^\top \frac{\partial (H+U)}{\partial x}^\top - \beta$, which coincides with the proposed controller.

By substituting the optimal control into the SHJB equation (40), the right hand side of the equation is reduced to

$$\begin{aligned}&\frac{\partial (H+U)}{\partial x}(J-R)\frac{\partial H}{\partial x}^\top - \frac{\partial (H+U)}{\partial x}^\top g\beta \\ &+\frac{1}{2}\operatorname{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial (H+U)}{\partial x}\right)^\top h h^\top\right\}.\end{aligned}$$

Since the condition for the SGCT (22) in Theorem 1 is satisfied, by eliminating $g\beta$ we have

$$\begin{aligned}\text{R.H.S. of Eq. (40)} = &-\frac{\partial (H+U)}{\partial x}(R+Q)\frac{\partial (H+U)}{\partial x}^\top \\ &-\frac{\partial (H+U)}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}\left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}\begin{bmatrix}\operatorname{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^1}{\partial x}\right)^\top h h^\top\right\}\\ \vdots \\ \operatorname{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^n}{\partial x}\right)^\top h h^\top\right\}\end{bmatrix}\right) \\ &+\frac{1}{2}\operatorname{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial (H+U)}{\partial x}\right)^\top h h^\top\right\}.\end{aligned} \tag{41}$$

Here we consider the dynamics $\mathrm{d}x = (J-R)\frac{\partial (H+U)}{\partial x}^\top \mathrm{d}t + h\,\mathrm{d}w$, and the coordinate transformation (3). Since the infinitesimal generator is invariant under the coordinate transformation, $\mathcal{L}(H+U) = \mathcal{L}\bar H$ holds, where $\mathcal{L}(H+U)$ is calculated along the above dynamics and $\mathcal{L}\bar H$ is calculated along the transformed dynamics by the transformation (3), respectively. $\mathcal{L}(H+U)$ and $\mathcal{L}\bar H$ are calculated as follows:

$$\begin{aligned}\mathcal{L}(H+U) = &\frac{\partial (H+U)}{\partial t} + \frac{\partial (H+U)}{\partial x}(J-R)\frac{\partial (H+U)}{\partial x}^\top \\ &+\frac{1}{2}\operatorname{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial (H+U)}{\partial x}\right)^\top h h^\top\right\}\end{aligned}$$

$$\mathcal{L}\bar{H} = \frac{\partial \bar{H}}{\partial \bar{t}} + \frac{\partial \bar{H}}{\partial \bar{x}}\left(\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x}(J-R)\frac{\partial H}{\partial x}^\top + \frac{1}{2}\begin{bmatrix} \mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^1}{\partial x}\right)^\top hh^\top\right\} \\ \vdots \\ \mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^n}{\partial x}\right)^\top hh^\top\right\}\end{bmatrix}\right) + \frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{H}}{\partial \bar{x}}\right)^\top \bar{h}\bar{h}^\top\right\}.$$

It follows from $\mathcal{L}(H+U) = \mathcal{L}\bar{H}$ and Eqs. (35) and (37) that

$$\frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial (H+U)}{\partial x}\right)^\top hh^\top\right\} = \frac{1}{2}\frac{\partial (H+U)}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1} \times \begin{bmatrix} \mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^1}{\partial x}\right)^\top hh^\top\right\} \\ \vdots \\ \mathrm{tr}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \Phi^n}{\partial x}\right)^\top hh^\top\right\}\end{bmatrix} + \frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{H}}{\partial \bar{x}}\right)^\top \bar{h}\bar{h}^\top\right\}. \quad (42)$$

From Eqs. (41) and (42), we have

$$\begin{aligned} \text{R.H.S. of Eq. (40)} &= -\frac{\partial (H+U)}{\partial x}(R+Q)\frac{\partial (H+U)}{\partial x}^\top \\ &\quad - \frac{\partial (H+U)}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}\frac{\partial \Phi}{\partial t} + \frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{H}}{\partial \bar{x}}\right)^\top \bar{h}\bar{h}^\top\right\} \\ &= -\frac{\partial (H+U)}{\partial x}\left[\frac{\partial \Phi}{\partial x}\right]^{-1}\frac{\partial \Phi}{\partial t} - \frac{\partial \bar{H}}{\partial \bar{t}} = -\frac{\partial (H+U)}{\partial t}, \quad (43) \end{aligned}$$

where the second equality comes from the assumption that the equality holds in the inequality (33) in Lemma 4, and the last equality comes from Eqs. (35) and (37). Eq. (43) implies that the proposed controller satisfies the SHJB equation with the value function $H+U$. ∎

V. Numerical examples

This section exhibits applications of the proposed stabilization method to a nonholonomic system in the presence of noise.

A. Time-invariant controller design

We apply Lemma 4 and Theorems 1 and 2 to control of a rolling coin to an invariant set in probability in the presence of noise. We consider the rolling coin on a horizontal plane [11], [26] depicted in Fig. 1. Let X-Y denote the orthogonal coordinates of the point of contact of the coin. Let q^1 be the heading angle of the coin, and (q^2, q^3) denote the position of the coin in the X-Y plane. Furthermore let p^1 be the angular velocity with respect to the heading angle, p^2 be the rolling angular velocity of the coin, u^1 and u^2 be the accelerations with respect to p^1 and p^2, respectively. Finally, let all the parameters be unity for simplicity, e.g., the radius of the coin and the moments of inertia. Then, this system is described by a SPHS of the form (2) with $q = (q^1, q^2, q^3)^\top$, $p = (p^1, p^2)^\top$, $x = (q^\top, p^\top)^\top$, $H(x) = (1/2)p^\top p$, $R(x) = 0_{5\times 5}$, $g(x) = (0_{2\times 3}, I_2)^\top$ and

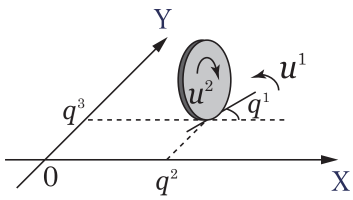

Fig. 1. Rolling coin

$$\begin{aligned} J(x) &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cos q^1 \\ 0 & 0 & 0 & 0 & \sin q^1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -\cos q^1 & -\sin q^1 & 0 & 0 \end{pmatrix} \\ h(x) &= \begin{pmatrix} 0_{3\times 2} \\ \mathrm{diag}\{h^1(x), h^2(x)\} \end{pmatrix} \\ y &= g(x)^\top \frac{\partial H(x)}{\partial x}^\top = p, \end{aligned} \quad (44)$$

where I_n represents the $n\times n$ identity matrix, and $h^1(x)$ and $h^2(x)$ represent functions for the noise port. For the details of nonholonomic Hamiltonian systems, see, e.g., [27], [28].

In [26], a stabilization technique of deterministic nonholonomic Hamiltonian systems is proposed, where a system is converted into a canonical form of nonholonomic Hamiltonian systems by a generalized canonical transformation with any potential function U such that it has the form $U((q^1)^2 + (q^2)^2, q^3)$ and is smooth and positive definite on $(q^1, q^2) \neq 0$ and satisfies

$$(q^1, q^2) \neq 0 \quad \Rightarrow \quad \frac{\partial U}{\partial q} \neq 0, \quad (45)$$

and with a special time-invariant coordinate transformation $\bar{x} = \Phi(x)$. Then convergence of the state to the following specified invariant set is achieved:

$$\Pi_0 := \left\{x \mid q^1 = q^2 = 0\ ,\ p^1 = p^2 = 0\right\}. \quad (46)$$

The purpose of this subsection is to let the state converge to the invariant set Π_0 in probability in the presence of noise by using the same coordinate transformation proposed in [26].

Since the Hamiltonian of the system (44) is positive semi-definite, let us transform the system into anther stochastic passive system by the SGCT with a potential function $U((q^1)^2 + (q^2)^2, q^3)$ and the following coordinate transformation $\bar{x} = \Phi(x)$ proposed in [26]:

$$\begin{aligned} \bar{q} &= \left(\Phi^1(x), \Phi^2(x), \Phi^3(x)\right)^\top = \left(\tan q^1, q^2, 2q^3 - q^2 \tan q^1\right)^\top \\ \bar{p} &= \left(\Phi^4(x), \Phi^5(x)\right)^\top = \left(\frac{p^1}{1+\tan^2 q^1}, p^2\sqrt{1+\tan^2 q^1}\right)^\top. \end{aligned} \quad (47)$$

In order to obtain the SGCT, we decide the other design parameters $\beta(x)$, $P(x)$ and $Q(x)$ according to Theorem1. The following choice satisfies Eq. (22): $P(x)=0_{5\times5}$ and

$$Q(x)=\left(\begin{array}{c|cc} 0_{3\times3} & \multicolumn{2}{c}{0_{3\times2}} \\ \hline 0_{2\times3} & Q_{44}(x) & Q_{45}(x) \\ & Q_{45}(x) & Q_{55}(x) \end{array}\right)=:\left(\begin{array}{c|c} 0_{3\times3} & 0_{3\times2} \\ \hline 0_{2\times3} & \tilde{Q}(x) \end{array}\right)$$

$$\beta(x)=\begin{pmatrix} \frac{\partial U}{\partial q^1}+p^1Q_{44}(x)+p^2Q_{45}(x) \\ \frac{\partial U}{\partial q^2}\cos q^1+\frac{\partial U}{\partial q^3}\sin q^1+p^1Q_{45}(x)+p^2Q_{55}(x) \end{pmatrix}$$

$$\alpha(x)=g^\top\frac{\partial U}{\partial x}^\top=\begin{pmatrix} 0&0&0&1&0\\0&0&0&0&1 \end{pmatrix}\left(\frac{\partial U}{\partial q_1},\frac{\partial U}{\partial q_2},\frac{\partial U}{\partial q_3},0,0\right)^\top$$
$$=0_{2\times1}, \tag{48}$$

where free parameters $Q_{44}(x), Q_{45}(x)$ and $Q_{55}(x)$ should be chosen so that $\tilde{Q}(x)$ in Eq. (48) becomes symmetric positive semi-definite. Furthermore, we derive another condition for $Q_{44}(x), Q_{45}(x)$ and $Q_{55}(x)$ from Lemma 4 for stochastic passivity. It follows from the inequality (33) that

$$\frac{1}{2}(h^1(x)^2+h^2(x)^2)$$
$$\le (p^1)^2Q_{44}(x)+2p^1p^2Q_{45}(x)+(p^2)^2Q_{55}(x). \tag{49}$$

If there exist functions $Q_{44}(x), Q_{45}(x)$ and $Q_{55}(x)$ for a noise port $h(x)$ such that they satisfy the condition (49), and let $\tilde{Q}(x)$ in Eq. (48) become symmetric positive semi-definite, the transformed system obtains stochastic passivity. From another viewpoint, the inequality (49) implies a condition for the noise port $h(x)$ under which the system can be stabilized based on stochastic passivity. Let us note that the condition (48) is independent of the choice of a potential function $U((q^1)^2+(q^2)^2,q^3)$. In what follows, we suppose that there exist functions $Q_{44}(x), Q_{45}(x)$ and $Q_{55}(x)$ such that the equality (49) holds and $\tilde{Q}(x)$ becomes symmetric positive semi-definite.

The transformed system is given by

$$\bar{J}(\bar{x})=\begin{pmatrix} 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&-\bar{q}^2&\bar{q}^1\\ -1&0&\bar{q}^2&0&-\bar{p}^2\bar{q}^1/(1+(\bar{q}^1)^2)\\ 0&-1&-\bar{q}^1&\bar{p}^2\bar{q}^1/(1+(\bar{q}^1)^2)&0 \end{pmatrix}$$

$$\bar{R}(\bar{x})=\left.\left(\begin{array}{c|cc} 0_{3\times3} & \multicolumn{2}{c}{0_{3\times2}} \\ \hline 0_{2\times3} & \cos^4 q^1Q_{44}(x) & \cos q^1Q_{45}(x) \\ & \cos q^1Q_{45}(x) & \dfrac{Q_{55}(x)}{\cos^2 q^1} \end{array}\right)\right|_{x=\Phi^{-1}(\bar{x})}$$

$$\bar{g}(\bar{x})=\begin{pmatrix} 0_{3\times2} \\ \mathrm{diag}\{1/(1+(\bar{q}^1)^2),\sqrt{1+(\bar{q}^1)^2}\} \end{pmatrix} \tag{50}$$

$$\bar{h}(\bar{x})=\left.\begin{pmatrix} 0_{3\times2} \\ \mathrm{diag}\{h^1(x)\cos^2 q^1, h^2(x)/\cos q^1\} \end{pmatrix}\right|_{x=\Phi^{-1}(\bar{x})}.$$

Eq. (50) implies that the transformed system has the form of (2). Since this system obtains stochastic passivity, it can be easily proven by Theorem 2 that with the unity feedback $\bar{u}=-\bar{y}$, the state $\bar{x}$ tends in probability to the largest invariant set whose support is contained in the locus $\mathcal{L}\bar{H}(\bar{x})=0$. Now let us show that this largest invariant set coincides with our target set Π_0 in (46). Since $Q_{44}(x), Q_{45}(x)$ and $Q_{55}(x)$ are chosen to satisfy the equality in the condition (49), the equation $\frac{1}{2}\mathrm{tr}\left\{\frac{\partial}{\partial\bar{x}}\left(\frac{\partial\bar{H}(\bar{x})}{\partial\bar{x}}\right)^\top\bar{h}(\bar{x})\bar{h}(\bar{x})^\top\right\}=\frac{\partial\bar{H}(\bar{x})}{\partial\bar{x}}\bar{R}(\bar{x})\frac{\partial\bar{H}(\bar{x})}{\partial\bar{x}}^\top$ holds. Then, the passivity of the transformed system yields

$$\mathcal{L}\bar{H}(\bar{x})=\bar{y}^\top\bar{u}. \tag{51}$$

Eq. (51) implies that we should show that the input/output nulling set coincides with Π_0. The following calculation requires that $\{\bar{x}\mid\bar{p}=0\}$ for the input/output nulling set: $0\equiv\bar{y}=\bar{g}^\top\frac{\partial\bar{H}}{\partial\bar{x}}^\top=\left((1+(\bar{q}^1)^2)\bar{p}^1,\bar{p}^2/\sqrt{1+(\bar{q}^1)^2}\right)^\top$. Under the condition $\{\bar{x}\mid\bar{p}=0\}$, we can calculate

$$\mathcal{L}_0L_{\bar{g}_1}\bar{H}(\bar{x})\Big|_{\bar{p}=0} = -(1+(\bar{q}^1)^2)\left(\frac{\partial U}{\partial\bar{q}^1}-\bar{q}^2\frac{\partial U}{\partial\bar{q}^3}\right)$$
$$\mathcal{L}_0L_{\bar{g}_2}\bar{H}(\bar{x})\Big|_{\bar{p}=0} = -\left(\frac{\partial U}{\partial\bar{q}^2}+\bar{q}^1\frac{\partial U}{\partial\bar{q}^3}\right)\Big/\sqrt{1+(\bar{q}^1)^2}.$$

From the conditions that $\mathcal{L}_0L_{\bar{g}_1}\bar{H}(\bar{x})\big|_{\bar{p}=0}\equiv0$ and $\mathcal{L}_0L_{\bar{g}_2}\bar{H}(\bar{x})\big|_{\bar{p}=0}\equiv0$, the input/output nulling set is obtained as $\left\{\bar{x}\;\middle|\;\frac{\partial U}{\partial\bar{q}}\bar{J}_{12}(\bar{q})=0,\ \bar{p}=0\right\}$, where

$$\bar{J}_{12}(\bar{q}):=\begin{pmatrix} 1&0&-\bar{q}^2\\0&1&\bar{q}^1 \end{pmatrix}^\top.$$

From the condition of the potential function U (see Eq. (45)) and the coordinate transformation Φ in (47), it can be easily calculated as

$$\left\{\bar{x}\;\middle|\;\frac{\partial U}{\partial\bar{q}}\bar{J}_{12}(\bar{q})=0,\ \bar{p}=0\right\}$$
$$=\left\{\bar{x}\mid\bar{q}^1=\bar{q}^2=0\ ,\ \bar{p}^1=\bar{p}^2=0\right\}$$
$$=\left\{x\mid q^1=q^2=0\ ,\ p^1=p^2=0\right\}\equiv\Pi_0. \tag{52}$$

From Theorem 2 and Eq. (52), we obtain the controller which achieves convergence to Π_0 as

$$u=-\beta(x)-(y+\alpha(x))$$
$$=-\begin{pmatrix} \frac{\partial U}{\partial q^1}+p^1(1+Q_{44}(x))+p^2Q_{45}(x) \\ \frac{\partial U}{\partial q^2}\cos q^1+\frac{\partial U}{\partial q^3}\sin q^1+p^1Q_{45}(x)+p^2(1+Q_{55}(x)) \end{pmatrix}. \tag{53}$$

The purpose of this subsection has been achieved.

Finally, let us show some simulation results. Here we consider the noise port as

$$h(x)=\begin{pmatrix} 0&0&0&k^1p^1&0\\0&0&0&0&k^2p^2 \end{pmatrix}^\top$$

and we set a potential function as $U=(1/2)q^\top q$, design parameters as $Q_{44}=(k^1)^2/2$, $Q_{45}=0$ and $Q_{55}=k_2^2/2$ with $k^1=k^2=15$ and the initial condition as $(q^1,q^2,q^3,p^1,p^2)=(0.1,0.4,0.2,0,0)$. We simulate a standard Wiener process in the same manner as in [20].

First, we consider a scenario where there exists noise, and a feedback controller designed by a deterministic method in [26] is applied. The deterministic controller corresponds to the controller (53) with $Q(x)=0$, and we verified that the objective is achieved with the controller without noise. Fig. 2 shows the motion of the coin in the X-Y plane and Fig. 3 shows the time responses of q and p, respectively. They imply that the behavior of the system seems unstable with the controller for the deterministic system in the presence of noise.

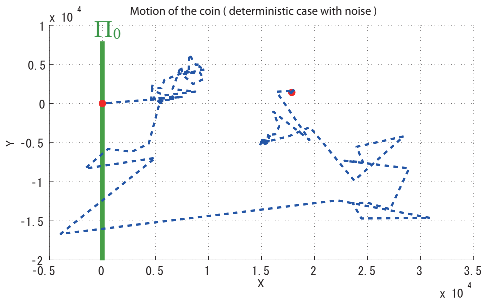

Fig. 2. Motion of the coin in the X-Y plane in the presence of noise with a deterministic controller

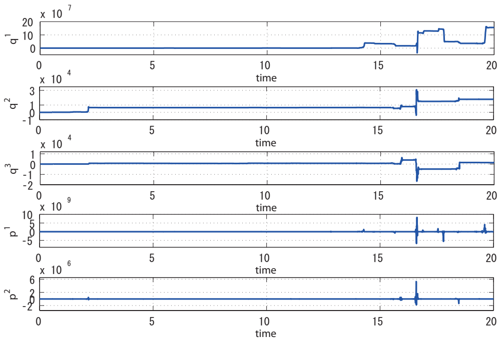

Fig. 3. Responses of q and p in the presence of noise with a deterministic controller

Finally, we consider a scenario where there exists the same noise as in the previous scenario and the feedback controller (53) designed by the proposed method is applied. Fig. 4 shows the motion of the coin in the X-Y plane and Fig. 5 shows the time responses of q and p, respectively. They imply that the proposed controller works well even in the presence of noise.

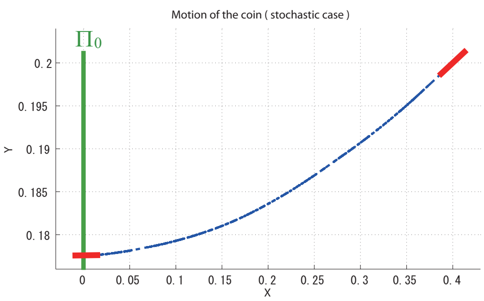

Fig. 4. Motion of the coin in the X-Y plane with the proposed controller with the same noise signal to the previous case

These simulation results demonstrate the effectiveness of the proposed framework.

Remark 5: Due to Brockett's condition [29], deterministic nonholonomic systems can not be asymptotically stabilized around a fixed point under any continuous static feedback controller. In the case of the stochastic system (44), it is easily

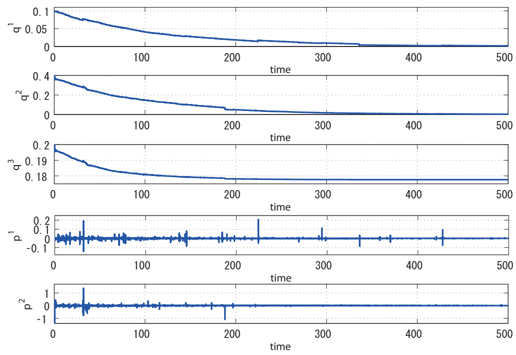

Fig. 5. Responses of q and p with the proposed controller with the same noise signal to the previous case

shown that the condition in Theorem 2 that $\bar{\Gamma}\cap\bar{\Pi}=\{0\}$ does not hold. Since $\mathcal{L}_0\bar{H}(\bar{x}) = \frac{\partial \bar{H}}{\partial \bar{x}}\bar{f}_0 + \frac{1}{2}\,\mathrm{tr}\left\{\frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{H}}{\partial \bar{x}}\right)^{\top}\bar{h}\bar{h}^{\top}\right\} = 0$, we have $\bar{\Gamma} = \mathbb{R}^5$. From the above argument, we have $\bar{\Pi} = \left\{\bar{x}\;\middle|\;\frac{\partial U}{\partial \bar{q}}\bar{J}_{12}(\bar{q}) = 0,\ \bar{p} = 0\right\}$. Therefore $\bar{\Gamma}\cap\bar{\Pi} \neq \{0\}$ is shown. In the literatures [26], [28], a discontinuous static feedback controller and a time-varying feedback controller which render the origin of a deterministic nonholonomic port-Hamiltonian system asymptotically stable, are proposed, respectively. In the next subsection, we design a time-varying feedback controller which renders the origin of the same system in (44) asymptotically stable in probability.

B. Time-varying controller design

In this subsection, we again consider the rolling coin in the presence of noise described in (44). In subsection V-A, we have designed a continuous static controller which makes the state converge to an invariant set in probability in the presence of noise. On the other hand, the literature [28] has proposed time-varying asymptotically stabilizing controllers for deterministic nonholonomic Hamiltonian systems. In this method, a special class of time-varying generalized canonical transformations is introduced in which a time-varying potential function $U(x,t)$ is parameterized by an arbitrary periodic function $\alpha(q,t)$ which is the parameter of the change of the output. The purpose of this subsection is to design a time-varying feedback controller which renders the origin asymptotically stable in probability based on the technique in [28].

First, we consider the following form of the new Hamiltonian:

$$\bar{H} = H + p^{\top}\alpha + \frac{1}{2}\alpha^{\top}\alpha + V(\bar{q}) = \frac{1}{2}(p+\alpha)^{\top}(p+\alpha) + V(\bar{q}),$$

where $\alpha(q,t)$ is any periodic odd function and an appropriate function $V(\bar{q})$ should be chosen so that $\bar{H}$ is non-negative in the new coordinate. Here we utilize the following functions which are the same as those in [28]:

$$\alpha(q,t) = \left(q^3 \sin t, 0\right)^{\top}, \quad V(\bar{q}) = \frac{1}{2}\bar{q}^{\top}K\bar{q}, \qquad (54)$$

where K is defined as $K := \mathrm{diag}\{k^1, k^2, k^3\}$ with appropriate positive numbers k^1, k^2 and k^3. Then let us construct a time-varying stochastic generalized canonical transformation with the following coordinate transformation utilized in [28]:

$$\bar{q} = \left(q^1 - q^3 \cos t, q^2, q^3\right)^\top \ , \quad \bar{p} = \left(p^1 + q^3 \sin t, p^2\right)^\top . \tag{55}$$

In order to obtain a time-varying SGCT, let us decide the other design parameters $\beta(x,t), P(x,t)$ and $Q(x,t)$ by utilizing Theorem1. The following choice satisfies Eq. (22):

$$P(x,t) = 0_{5\times 5} \ , \quad Q(x,t) = \left(\begin{array}{c|c} 0_{3\times 3} & 0_{3\times 2} \\ \hline 0_{2\times 3} & \begin{matrix} Q_{44}(x,t) & 0 \\ 0 & Q_{55}(x,t) \end{matrix} \end{array}\right) \tag{56}$$

$$\begin{aligned} \beta(x,t) =& (q^3 \cos t + k^1(q^1 - q^3 \cos t) + p^2 \sin t \sin q^1 \\ & + (p^1 + q^3 \sin t) Q_{44}(x,t), \\ & - k^1(q^1 - q^3 \cos t) \sin q^1 \cos t + k^2 q^2 \cos q^1 \\ & + k^3 q^3 \sin q^1 + p^2 Q_{55}(x,t))^\top, \end{aligned} \tag{57}$$

where the free parameters $Q_{44}(x,t)$ and $Q_{55}(x,t)$ should be chosen so that $Q(x,t)$ in Eq. (56) becomes symmetric positive semi-definite. Furthermore, in order to obtain stochastic passivity, let us derive another condition for $Q_{44}(x,t)$ and $Q_{55}(x,t)$ from Lemma 4. It follows from the inequality (33) that

$$\begin{aligned} & \frac{1}{2}(h^1(x,t)^2 + h^2(x,t)^2) \\ & \qquad \leq (p^1 + q^3 \sin t)^2 Q_{44}(x,t) + (p^2)^2 Q_{55}(x,t). \end{aligned} \tag{58}$$

In what follows, we suppose that there exist functions $Q_{44}(x,t)$ and $Q_{55}(x,t)$ such that the equality in the inequality (58) holds and $Q(x,t)$ in Eq. (56) becomes symmetric positive semi-definite.

The transformed system is given by $\bar{g}(\bar{x}) = (0_{2\times 3}, I_2)^\top$, $\bar{y} = \bar{p}$ and

$$\begin{aligned} &\bar{J}(\bar{x},\bar{t}) = \\ &\left.\begin{pmatrix} 0 & 0 & 0 & 1 & -\sin q^1 \cos t \\ 0 & 0 & 0 & 0 & \cos q^1 \\ 0 & 0 & 0 & 0 & \sin q^1 \\ -1 & 0 & 0 & 0 & \sin q^1 \sin t \\ \sin q^1 \cos t & -\cos q^1 & -\sin q^1 & -\sin q^1 \sin t & 0 \end{pmatrix}\right|_{\substack{x = \Phi^{-1}(\bar{x},\bar{t}) \\ t = \bar{t}}} \\ &\bar{R}(\bar{x},\bar{t}) = Q(x,t)\Big|_{\substack{x = \Phi^{-1}(\bar{x},\bar{t}) \\ t = \bar{t}}} \qquad (59) \\ &\bar{h}(\bar{x}) = \left.\begin{pmatrix} 0_{3\times 2} \\ \mathrm{diag}\{h^1(x,t), h^2(x,t)\} \end{pmatrix}\right|_{\substack{x = \Phi^{-1}(\bar{x},\bar{t}) \\ t = \bar{t}}} . \end{aligned}$$

Eq. (59) implies that the transformed system has the form of (2). Since this system obtains stochastic passivity, it can be easily proven by Theorem 3 that $\lim_{\bar{t}\to\infty} \bar{y} = 0$ almost surely with the unity feedback $\bar{u} = -\bar{y}$. Generally, Theorem 3 only guarantees the convergence of the output. However, in this case, we can show that the unity feedback also renders the origin of the system (44) asymptotically stable almost surely. Let $\bar{u} = \bar{y} = \bar{p} \equiv 0$ of the system (59). Then it follows from Eq. (55) and the condition (58) that $h(x,t) \equiv 0_{5\times 2}$. The literature [28] has proven that the transformed system (59) has zero-state observability with respect to x without noise. These two facts prove the claim. Eventually, we obtain the following time-varying feedback controller which renders the origin asymptotically stable almost surely as

$$u = -\beta(x,t) - (y + \alpha(x,t)) \tag{60}$$

(see, Eqs.(54) and (57) for $\alpha(x,t)$ and $\beta(x,t)$).

Finally, let us show some simulation results. Here we consider the noise port as $h^1(x,t) = 0$ and $h^2(x,t) = k^2 p^2$ and we set a matrix K in (54) as $K = \mathrm{diag}\{1,1,1\}$, design parameters as $Q_{44} = 0$ and $Q_{55} = k_2^2/2$ with $k^2 = 8$ and the initial condition as $(q^1, q^2, q^3, p^1, p^2) = (0, 0, 1.0, 0, 0)$. Since convergence is slow and oscillatory in the case of equipping time-varying feedback controllers [30], [28], we execute subsequent simulations with different initial condition and design parameters from those in subsection V-A.

First, we consider a scenario where there exists noise, and a feedback controller designed by a deterministic method in [28], which corresponds to the controller (60) with $Q(x,t) = 0_{5\times 5}$, is applied. Although the convergence is slow and oscillatory, we verified that the objective is achieved with the controller without noise. Fig. 6 shows the motion of the coin in the X-Y plane and Fig. 7 shows the time responses of q and p. They imply that the behavior of the system seems unstable with the controller for the deterministic system in the presence of noise.

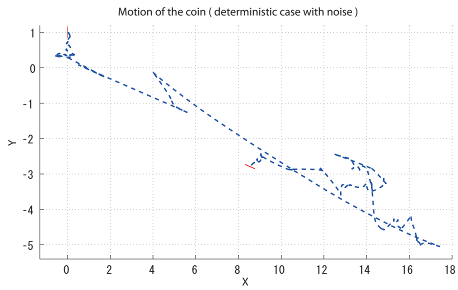

Fig. 6. Motion of the coin in the X-Y plane in the presence of noise with a time-varying deterministic controller

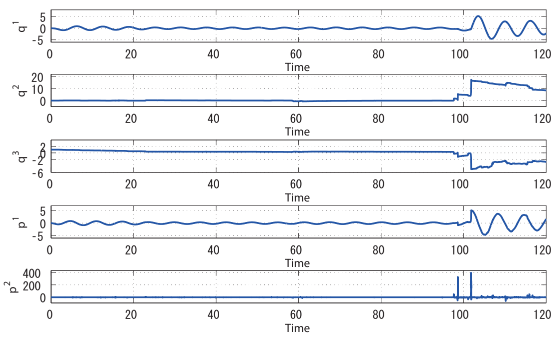

Fig. 7. Responses of q and p in the presence of noise with a time-varying deterministic controller

Finally, we consider a scenario where there exists the same noise as in the previous scenario and the feedback controller (60) designed by the proposed method is applied. Fig. 8 shows the motion of the coin in the X-Y plane and Fig. 9 shows the time responses of q and p. They imply that the proposed controller works well even in the presence of noise, although the convergence is slower and more oscillatory than the case without noise.

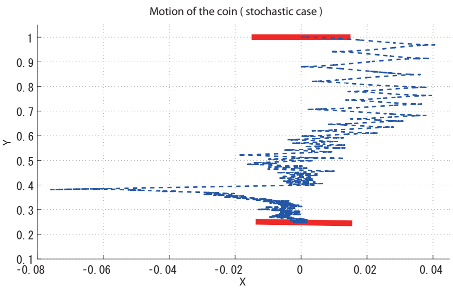

Fig. 8. Motion of the coin in the X-Y plane with the proposed time-varying controller with the same noise signal to the previous case

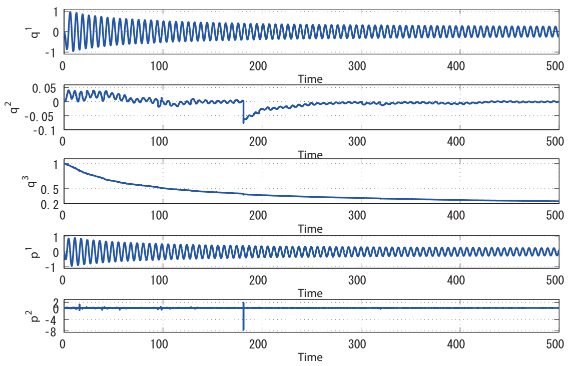

Fig. 9. Responses of q and p with the proposed time-varying controller with the same noise signal to the previous case

Those simulation results demonstrate the effectiveness of the proposed method.

VI. Conclusion

This paper has introduced stochastic port-Hamiltonian systems (SPHS's) and has clarified some of their properties. First, we have shown a necessary and sufficient condition for the SPHS structure to be preserved under coordinate transformations. Second, we have derived a condition to achieve stochastic passivity of the system. Third, we have introduced stochastic generalized canonical transformations (SGCT's). Fourth, we have provided a condition that the transformed system by the SGCT becomes stochastic passive and have proposed a stabilization method based on both stochastic passivity and SGCT's. Although stabilization of general nonlinear stochastic systems is very difficult, we focus on an important class of them and can provide a systematic procedure with explicit conditions for a stabilizing controller. Finally, numerical simulations demonstrate the effectiveness of the proposed method.

Since the condition (22) is a second order partial differential equation, we did not investigate how to solve it yet. However, since it is linear, we now try to parameterize the solution in the case of mechanical systems. A systematic solution to the condition is our future work.

In the proposed SPHS (2) as well as the deterministic Hamiltonian system (1), the output is defined so that the input and the corresponding output are the effort and the flow variables, whose product is always power. This definition enables the system to possess an intrinsic property such as passivity. If an extra structure is added to the output and/or the noise port, it might induce a useful property between the noise and the system such as the noise-to-state stability concept proposed in [31]. Further investigation of another possible structure of the output and noise port is also our future work.

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